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Contrariety and Subcontrariety: The Anatomy of Negation (with Special Reference to an Example of J.-Y. Béziau)

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ABSTRACT. We discuss aspects of the logic of negation bearing on an issue raised by Jean-Yves Béziau, recalled in §1. Contrary- and subcontrary-forming operators are introduced in §2, which examines some of their logical behaviour, leading on naturally to a consideration in §3 of dual intuitionistic negation (as well as implication), and some further operators related to intuitionistic negation. In §4, a historical explanation is suggested as to why some of these negation-related connectives have attracted more attention than others. The remaining sections (§§5, 6) briefly address a question about a certain notion of global contrariety and the provision of Kripke semantics for the various operators in play in our discussion.

1. Introduction and Recapitulation

This note supplements the discussion of [1] provided by [9] – which it would be helpful, though not essential, to have at hand – in which several issues concerning contrariety and subcontrariety arose that could not, for reasons of length, be dealt with within the confines of [9]. We begin with a summary of the main point of [1], expressed in the notation and terminology of [9]. With respect to a class of valuations (assignments to the formulas of a propositional language of the truth-values T, F), two formulas are contraries if no valuation in the class assigns the value T to both of them, and subcontraries if no such valuation assigns the value F to both of them. Derivatively, two formulas are (sub)contraries according to the logic – whether conceived of as a set of formulas or as a consequence relation (or even as a *generalized* or “multiple conclusion” consequence relation) determined by a class of valuations when they are (sub)contraries with respect to that class, in the sense just defined. If we extend the classical logic of implication (notation: →) by adding a 1-ary connective ¬ to its language and just enough by way of logical principles to the logic to secure that for any formula A, A and ¬A are contraries, then we do not have a complete formalization of the implication-negation fragment of classical logic, identifying ¬ as negation, since a formula and its negation are not also subcontraries according to the logic thus extended.[1] But if we then proceed to define a new 1-ary connective, ~, say, by means of (Def.1~):

(Def.1~) $\sim A = A \to \neg A$

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[1] Note that one a certain tradition – on which see, e.g., [8] – contrariety and subcontrariety are mutually exclusive, though this is not so for the way the notions have been defined here.

then we find that A and ~A are not only contraries but now also subcontraries for any formula A, so the full classical logic of implication and negation can after all be recovered once we identify negation as expressed by the defined ~ rather than the primitive ¬. Another definition (from Section 3 of [9]) which would have the same effect, reminiscent of the definition of negation in terms of a *Falsum* constant is the following:

(Def.2~) $\sim A = A \to \neg(A \to A)$.

These examples, the first of which is what we have in mind – e.g. in the title of this note – as Béziau's (main) example, are somewhat puzzling in that we don't know whether to say that the "(material) → + (merely contrary-forming) ¬" logic is weaker than classical logic or not: in one sense, we want to say that it is weaker since ¬ lacks some of the principles governing classical negation, while in another we want to say that it is stronger, since the logic contains the whole of classical (implication–negation) logic in the connectives → and ~, as well as having the further ingredient ¬ corresponding to no primitive or defined connective of classical logic. If we say both things at once, we will end up regarding the situation as paradoxical – a suggestion taken seriously in [1] though contested in (especially Section 5 of) [9]. This question of paradoxicality does not concern us here, however. [9] also points out that if we had started not with the classical but with the intuitionistic logic of implication, and added a contrary-forming operator ¬, then what (Def.1~) and (Def.2~) define is not an operator that yields a formula that is simultaneously a contrary and a subcontrary of what it applies to, but rather yields a formula which is the (deductively) weakest contrary of that formula – the usual significance of intuitionistic negation. (In Section 4 we shall have occasion to consider also *dual* intuitionistic negation.) The phrase "simultaneously a contrary and a subcontrary" makes perfect sense given the way contrariety and subcontrariety have been defined (in the opening paragraph above), though it goes against a tradition which would see these terms as incompatible – on which tradition, see, e.g., [8].

2. Contrarizing and Subcontrarizing Operators

The following further notation – and from now on the only time we encounter "~" is when (Def.1~) or (Def.2~) is cited – will assist us in treating these issues. To suggest the (admittedly non-Greek) words 'contrary' and 'subcontrary', we use κ and σ to denote 1-ary connectives for which, given any formula A, κA and σA are respectively a contrary of A and a subcontrary of A, whether understood as relative to a logic conceived of as consequence relation, or as a generalized consequence relation, or conceived of as a set of formulas.[2] In any of these cases, to simplify several points, we generally presume that → is also present and that it behaves intuitionistically: see the description of IL and the associated consequence relation $\vdash_{IL}$ in Section 1 of [9]. (CL and $\vdash_{CL}$ are the classical analogues, with $\Vdash_{CL}$ for the corresponding generalized consequence relation in this case. Except in asides, we will not discuss the treatment by way of generalized consequence

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[2] See note 6 of [4] for a reply to the criticism by Geach (of Storrs McCall) that since there is no such thing as *the* contrary of a statement, the use of any such notation as "κA" is confused (and similarly, *mutatis mutandis*, with subcontraries) – though the present treatment of contrariety does not coincide with that suggested in that paper (see note 21 in [9]).

relations.) For κ and σ to merit the above description as contrary and subcontrary forming operators, we lay down axiom-schemata affording them the required properties:

(Aκ) $A \to (\kappa A \to B)$

(Aσ) $(A \to B) \to ((\sigma A \to B) \to B)$.

Using intuitionistically acceptable principles governing $\to$ (such as (A1) and (A2) from Section 1 of [9]), from (Aκ) and (Aσ) one can prove

($\kappa\sigma$) $\kappa A \to \sigma A$.

(First put σA for B in (Aκ); then note the provability of $\sigma A \to (\kappa A \to \sigma A)$. *Modus Ponens* applied twice then delivers ($\kappa\sigma$) from the relevant instance of (Aσ).)

In terms of consequence relations, adapting (5) and (6) from Section 1 of [9], we have the principles

$(\kappa)_{\vdash}$ $A, \kappa A \vdash B$

$$(\sigma)_{\vdash} \quad \frac{\Gamma, A \vdash B \qquad\qquad \Gamma, \sigma A \vdash B}{\Gamma \vdash B}$$

In the case of $(\sigma)_{\vdash}$, we use the 'rule notation' to indicate that whenever the $\vdash$-statements above the line are satisfied for a given set of formulas Γ and formulas A, B, so is that below the line. (See note 1 of [9].) Again one verifies that for any consequence relation $\vdash$ satisfying $(\kappa)_{\vdash}$, and $(\sigma)_{\vdash}$, we also have

$(\kappa\sigma)_{\vdash}$ $\kappa A \vdash \sigma A$

To see this, put $\Gamma = \{\kappa A\}$, $B = \sigma A$, and the left-hand premiss for an appeal to $(\sigma)_{\vdash}$ is provided by $(\kappa)_{\vdash}$, and we get the right-hand premiss for nothing (i.e., just because $\vdash$ is a consequence relation); the conclusion of this application of $(\sigma)_{\vdash}$ is then $(\kappa\sigma)_{\vdash}$.[3]

The fact that we had to use a non-empty Γ in the derivation just sketched may prompt the worry that what $(\sigma)_{\vdash}$ tells us is more than that A and σA are subcontraries: that they are what we might call "subcontraries in the presence of arbitrary side-formulas", the side-formulas being collected into the set Γ. Indeed for an arbitrary consequence relation $\vdash$ (whose language is closed under σ) the general form of $(\sigma)_{\vdash}$ above cannot be derived from the special $\Gamma = \varnothing$ case of the rule, though we shall not prove this here, and the derivation of $(\kappa\sigma)_{\vdash}$ from $(\kappa)_{\vdash}$ and this restricted form of $(\sigma)_{\vdash}$ will not go through. Given the additional assumption that $\vdash$ is finitary ($\Gamma \vdash B$ always implying that $\Gamma_0 \vdash B$ for some finite $\Gamma_0 \subseteq \Gamma$, that is) and satisfies the principles of intuitionistic logic for $\to$ (given as (10) in Section 1 of [9]), however, the restricted form of $(\sigma)_{\vdash}$ suffices for the derivability

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[3] The matter would be even simpler using generalized consequence relations *à la* (3)–(4) in Section 1 of [9], where from the contrariety condition $A, \kappa A \Vdash \varnothing$ and the subcontrariety condition $\varnothing \Vdash A, \sigma A$, the desired conclusion, in the form $\kappa A \Vdash \sigma A$ follows by an appeal to the "Cut" condition on generalized consequence relations – with A as the Cut-formula. (The "$\Vdash$" notation is used in [9] to indicate that we are dealing with a generalized consequence relation.).

of the general form, since we can successively move the elements of Γ to the right as antecedents of an implication, apply the restricted rule, and then return them to the left.[4] The simplest guarantee of the derivability of the general form from the special case is the presence of disjunction with its accustomed logical powers. The special $\Gamma = \varnothing$ case yields $\vdash A \vee \sigma A$, which suffices to deliver the conclusion (with Γ arbitrary) of $(\sigma)_{\vdash}$ from its premisses.

A similar issue about side-formulas arises in our axiomatic presentation also. One might consider as a more deserving contender for giving σ the status of a subcontrary-forming operator than the axiom schema $(A\sigma)$ above, rather the following *rule*:

$$(R\sigma)\quad \frac{\vdash A \to B \qquad\qquad \vdash \sigma A \to B}{\vdash B}$$

Clearly, given $(A\sigma)$, $(R\sigma)$ is derivable – by two applications of *Modus Ponens*. The interesting question is over the converse derivability. For this, we observe that both (1) and (2) are provable using only IL for $\to$:

$$A \to ((A \to B) \to ((\sigma A \to B) \to B)) \tag{1}$$

$$\sigma A \to ((A \to B) \to ((\sigma A \to B) \to B)) \tag{2}$$

and from these $(R\sigma)$ delivers the conclusion $(A\sigma)$. Similar rules-*vs.*-axioms questions arise about $(A\kappa)$ and $(\kappa)_{\vdash}$, but we defer consideration of them until after some discussion of the contrast between intuitionistic and classical logic over contraries and subcontraries.

Subcontrariety in intuitionistic logic itself is a rather degenerate affair, since – to stick with the sets-of-formulas conception of logics – whenever formulas A and B are subcontraries in the sense (3) (= (8) from Section 1 of [9]) is IL-provable for all C:

$$(A \to C) \to ((B \to C) \to C) \tag{3}$$

then either A is provable or else B is, making either the first or the second antecedent of (3) redundant. (This is a simple reformulation of the Disjunction Property for IL. Note that the extension of IL with dual intuitionistic negation and implication – considered presently – lacks this property.) So, by contrast with the CL case, when A and B are subcontraries either A and D are subcontraries for all formulas D or else B and D are subcontraries for all D. This, incidentally, draws attention to a simple demonstration that adding σ as a new connective to IL, with axiom $(A\sigma)$, produces a conservative extension, since we already have an endogenous candidate to serve as the interpretation of σA for any A, namely $A \to A$ (or the *Verum* constant $\top$, if available). One can even add 'non-degenerate' subcontrarizers to IL conservatively, the best known example being the dual intuitionistic negation of C. Rauszer and others, which applied to A yields the *strongest* subcontrary of A, just as the usual intuitionistic negation yields the weakest contrary of A, a point

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[4] More accurately: we can perform the shunting operation just described upon the elements of $\Gamma_0 \cup \Gamma_1$, where Γ_0 is a finite subset of Γ securing the left-premiss of the unrestricted rule and Γ_1 a finite subset of Γ securing the right premiss.

mentioned in our opening paragraph and to which we return in the following section.[5] What has to be resisted, from an intuitionistic standpoint, is rather any confusion of a subcontrarizer with a contrarizer. ($\kappa\sigma$) gives us a one-way implication already, so the issue is over the converse, which would render κ and σ equivalent

($\sigma\kappa$) $\quad \sigma A \to \kappa A.$

The intuitionist's position is that while it is all right to have a 1-ary connective which turns a formula into a contrary of that formula, as well as a 1-ary connective which turns a formula into a subcontrary (in view of the preceding remarks about conservativity[6]), classical logic errs in assuming that it is all right to suppose that a single connective (the κ-*alias*-σ conflation, given ($\sigma\kappa$)) can play both of these roles at once. With only IL-acceptable implicational principles, and the governing axioms ($A\sigma$) and ($A\kappa$), we show as follows that the conflating principle ($\sigma\kappa$) would render Peirce's Law (= (A3) from Section 1 of [9]) provable.

From ($A\sigma$), by putting A for B we can derive

$$(\sigma A \to A) \to A \tag{4}$$

We continue to assume familiarity with simple deductive moves in intuitionistic logic. (In the present case, for example, we trust the reader to know that the antecedent resulting from the substitution just indicated, namely $A \to A$, is provable from (A1) and (A2) from Section 1 of [9], so that its consequent, given as (4), may be detached by *Modus Ponens*.) From ($A\kappa$) and ($\sigma\kappa$), we get

$$\sigma A \to (A \to B) \tag{5}$$

and from this by 'Suffixing' a consequent A:

$$((A \to B) \to A) \to (\sigma A \to A) \tag{6}$$

from which, together with (4), by 'Transitivity', we obtain Peirce's Law (= (A3) in [9]). Thus by conflating a contrary-forming operator with a subcontrary-forming operator, we extend positive implication nonconservatively (and in fact, all the way to classical implication). Such a remark hardly counts as news, but it may be of interest to see how the double life led by classical negation, acting simultaneously as κ and σ, leads to this result, especially as the above derivation nowhere uses a κ or σ within the scope of a κ or σ, by contrast with the more familiar double negation routes from intuitionistic to classical logic.

Digression. In fact there are some interesting embedding interactions for κ and σ themselves, such as the following repercussions of $(\kappa)_{\vdash}$ and $(\sigma)_{\vdash}$: for all A, we have $\kappa\sigma A \vdash A$ and $A \vdash \sigma\kappa A$. However, we confine our treatment of the 'anatomy of negation' to issues more closely tied to Béziau's example and its intuitionistic analogue, except for the following remarks. First, we note

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[5] See Section 4 of [6] for references to Rauszer's work – to which the more recent treatment in [3] should be added – as well as the debate over the legitimacy of dual intuitionistic negation ('Brouwer negation', in her terminology) as a possible 'new intuitionistic connective'. Kripke semantics for these connectives may be found at the end of the present note.

[6] This is an oversimplification, since more may be demanded of a new connective governed by certain rules (for it to count as intuitionistically intelligible) than that the rules yield a conservative extension: see the discussion and references in Sections 3 and 4 of [6].

that in the basic logic of contrariety and subcontrariety with $(\kappa)_{\vdash}$ and $(\sigma)_{\vdash}$ or with (Aκ) and (Aσ), any principle derivable with a schematic letter and the result of prefixing κ to that letter, ‘A’, say, can have all occurrences of κA interchanged with occurrences of A and the resulting principle is again derivable, and likewise with σA and A. This is just a way of saying that the relation *(sub)contrary of* is symmetric. For example, the principles recently mentioned, to the effect that $\kappa\sigma A \vdash A$ and $A \vdash \sigma\kappa A$, arise from $(\kappa\sigma)_{\vdash}$ by just such interchanges. Secondly, since κA has the logical properties of an arbitrary contrary of A, and σA those of an arbitrary subcontrary, once we have established a principle in the basic logic we know it will hold when κA (σA) is replaced by a particular contrary (subcontrary) of A. For example, from (Aσ), putting “σA” for “A” and “A” for “B”, we get (i) $(\sigma A \to A) \to A$, and thus by the interchange principle just mentioned, also (ii) $(A \to \sigma A) \to \sigma A$, which, $\neg A$ being a subcontrary of A in classical logic, appeared with $\neg$ in place of σ in Section 1 of [9] as (A5)′ and (A6)′ respectively. More interestingly, since, again classically, we have $A \to B$ as a subcontrary of A, putting this for σA in (i) and (ii) give Peirce’s Law and the Law of Contraction, respectively. Since in each of these cases, the (ii) form but not the (i) form is already intuitionistically acceptable, one might wonder if there are any instances in which $(A \to B) \to B$ and $(B \to A) \to A$ are both IL-provable without A or B itself being provable – making A, B degenerate subcontraries. There are such cases. In fact, the two implicational formulas mentioned here are, as is observed in [7], IL-provable if and only if A and B are respectively equivalent to $C \to D$ and $D \to C$ for some formulas C, D. ***End of Digression***.

3. Weakest Contraries and Strongest Subcontraries

One may wonder at this point whether a converse to the derivation preceding the Digression at the end of the previous section is also possible: that is: can ($\sigma\kappa$) in turn be derived (with the aid of (Aσ), (Aκ), and the classical logic of implication? The answer is negative, as one sees by interpreting σA and κA, for any formula A, as respectively $\top$ and $\bot$. This would make σA the logically weakest subcontrary of A (implied by every other subcontrary) and κA the strongest contrary (implying all the rest). Since this renders the 1-ary connectives κ and σ essentially 0-ary,[7] there has naturally been greater interest in stabilizing them by means of the reverse procedure: into connectives forming the weakest contrary and the strongest subcontrary of what they attach to.[8] As we have already mentioned, when the background implicational logic is weakened to that of IL, this gives, in the former case, the familiar intuitionistic negation and in the latter the less familiar dual intuitionistic negation. For the more familiar case, we can revisit the intuitionistic analogue – recalled in Section 1 above – of Béziau’s example, from Section 3 of [9], where κ appeared as “$\neg$”,

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[7] Thereby ‘globalizing’ them, in effect, to allude to terminology introduced in the second-last paragraph of this section.

[8] The talk of stabilization here is to be understood as the provision of logical properties sufficient to characterize uniquely (to within equivalence) a connective possessing the properties in question. Clearly, subjecting κ and σ merely to the conditions recorded in (Aκ) and (Aσ) does not do this. That is, to illustrate with the former case, we can lay down for two 1-ary connectives κ_1 and κ_2 that each should satisfy the condition for all formulas A, B, that $\vdash A \to (\kappa_1 A \to B)$ and $\vdash A \to (\kappa_2 A \to B)$ without its following that for all A $\vdash \kappa_1 A \to \kappa_2 A$ (and conversely). This is not quite the same point as was mentioned in note 2, since the same is true of the necessity operator in any conventional treatment of modal logic, and Geach has never complained that there is no such thing (for a given A) as *the* proposition that necessarily A.

by noting that for κA as a contrary of A (i.e. governed by (Aκ)), A → κA automatically becomes the strongest contrary of A. It is a contrary of A, for the reasons reviewed there (as well as in Section 2 of [9]), and if we suppose that B is also a contrary of A, i.e. that A → (B → C) is provable for all C, then in particular this holds for C = κA, so by permuting antecedents we have B → (A → κA), so A → κA (*alias* ~A from the earlier discussion in [9]) becomes the weakest contrary of A, provably implied by any other. For subcontrariety the situation is more complicated in several respects. In the first place we need to bring disjunction into the picture; secondly, we need to consider Rauszer's dual intuitionistic connectives, written here as $\neg_d$ and $\to_d$, for dual intuitionistic negation and dual intuitionistic implication, respectively, governed by the two-way rules (i.e., upward and downward transitions licensed):

$$\frac{\vdash A \vee B}{\neg_d A \vdash B} \qquad\qquad \frac{C \vdash A \vee B}{C \to_d A \vdash B} \qquad (7)$$

The rule on the right will not concern us for the moment, though we pause to note that a connective $\neg_d$ satisfying that on the left is definable in terms of a $\to_d$ satisfying this rule by taking $\neg_d A$ as $\top \to_d A$, where $\top$ is a *Verum* constant (or just use A → A in its place). The connective $\to_d$ is, as Curry has remarked (see [2], p.144, where an algebraic analogue is discussed under the name of subtraction and with the notation "–" rather than "$\to_d$"), not really the dual of implication but the dual of *converse* implication, which has the effect in classical logic (as we shall see below) of making $A \to_d B$ equivalent to the *negation* of A → B.

For the upward direction of the two-way rule on the left of (7), we could continue with our usual policy of using a new schematic letter "C" and replacing the *bona fide* disjunction A ∨ B with the 'deductive disjunction' (A → C) → ((B → C) → C) which has been our official formulation of subcontrariety (for A and B here). Of course if we are wanting (7) for the axiomatic approach we should move the formulas appearing on the left of the "⊢" over to the right as antecedents of implications (implications with →, that is). But we cannot do this for the downward direction, as with (8), in which we have made the adjustment just described,

$$\frac{\vdash (A \to C) \to ((B \to C) \to C)}{\vdash \neg_d A \to B} \qquad (8)$$

since the premiss now hypothesizes merely the provability for *some* formula C of the formula above the line, and we required, in order to encode the disjunction using only "→", that this hold for *all* formulas C. The well known definition of disjunction in intuitionistic second-order propositional logic has the effect of binding this free variable with a universal quantifier whose scope is just the premiss of (8), as in the downward direction of the $\neg_d$-rule (7). (There is a similar need to freeze a universal quantifier so that its scope is restricted to a premiss when we have a primitive *Falsum* ⊥ and want a rule for passing between Γ ⊢ A → ⊥ and Γ ⊢ ¬A, in that while in the backward direction we can just use a new schematic letter in place of $\bot$, for the forward direction this would not have the desired effect, turning the premiss into the hypothesis that for some C, we have $\Gamma \vdash A \to C$.) The upward and downward directions of the above $\neg_d$-rule also differ in the following respect. The upward direction is equivalent to its more general formulation with side-formulas on the left, while the downward direction is not. Indeed the upward direction can be replaced without loss by the schema $\vdash A \vee \neg_d A$, asserting the subcontrariety of any formula and its dual intuitionistic negation (thereby attesting to the already mentioned failure of the Disjunction Property for this extension of intuitionistic logic). Since we have already introduced the connective σ with this behaviour, it is worth introducing separately a connective for the 'strongestness' side of the picture. We write the new operator as $\tilde{\sigma}$, to be governed by the rule (9), corresponding to the downward direction of the $\neg_d$-rule (7), though reformulated so as to be applicable in the axiomatic setting:

$$\frac{\vdash A \vee B}{\vdash \tilde{\sigma} A \to B} \tag{9}$$

(9) does not say that $\tilde{\sigma} A$ is a subcontrary of A, only that it is at least as strong as any such subcontrary (schematically represented by B). Thus in particular the implication, taking σA for B to get a provable premiss for the rule, $\tilde{\sigma} A \to \sigma A$ is provable, and if we added the converse implication as a further principle, we would render σA and $\tilde{\sigma} A$ equivalent and have on our hands simply two new notations for $\neg_d A$.

Similarly in the case of contrariety and ordinary intuitionistic negation $\neg$. Let us supplement our contrariety operator κ with a 'weakestness' companion $\tilde{\kappa}$. In the same way as we helped ourselves to $\vee$ in the preceding discussion of subcontrariety, we presume here the presence of an all-implying $\bot$, so we can think of κ as governed by $A \to (\kappa A \to \bot)$, or equivalently $\kappa A \to (A \to \bot)$ and we want $\tilde{\kappa} A$ to be something which is provably implied by any contrary, B, say, of A:

$$\frac{\vdash B \to (A \to \bot)}{\vdash B \to \tilde{\kappa} A} \tag{10}$$

This makes $\tilde{\kappa} A$, not necessarily a contrary of A, but a formula which is weaker than any contrary of A. We can replace (10) without loss by the conclusion we should obtain on putting $A \to \bot$ for B in the premiss, so κA and $\tilde{\kappa} A$ now satisfy the two principles

$$\vdash \kappa A \to (A \to \bot) \qquad \textit{and} \qquad \vdash (A \to \bot) \to \tilde{\kappa} A \tag{11}$$

and hence we have $\kappa A \to \tilde{\kappa} A$ provable (compare the earlier $\tilde{\sigma} A \to \sigma A$) and if we add the converse implication we identify κ with $\tilde{\kappa}$ and have on our hands two notations for (intuitionistic) $\neg$. In Section 1 of [9], we remarked *à propos* of the *Reductio ad Absurdum* schema (A6), here repeated as (12)

$$\vdash (A \to C) \to ((A \to \neg C) \to \neg A), \tag{12}$$

that we had failed to separate out two roles played by ¬ here. The successive antecedents embody the hypothesis that A implies every formula only in virtue of the fact that C and ¬C are contraries, so we are having to presume that fact (itself laid down in a separate axiom – (A4) in Section 1 of [9]), which amounts to what we have just written as $A \to \bot$, before we go on to say that from this hypothesis ¬A follows. The two roles could be separated out with the current notation, as above, by putting κ for the first occurrence of ¬ in (12) and $\tilde{\kappa}$ for the second.

Dissatisfaction may be felt over another aspect of (12) as an embodiment of the idea that ¬A is to the weakest contrary of A, since it suppresses the idea of contrariety as a relation. We want to say that if A and B are contraries then B implies ¬A (or, in the refined notation, $\tilde{\kappa}A$), and if we are to use, after the manner of (12) the successive implication by A of a formula C and its negation ¬C (refined notation: κC) as a way of recording contrariety, a direction formulation of the hypothesis that A and B are contraries, so a direct formulation of this would be given rather by the two antecedents $A \to (B \to C)$ and $(A \to (B \to \neg C)$ of (13), here presented in the 'unrefined' notation:

$$\vdash (A \to (B \to C)) \to ((A \to (B \to \neg C) \to (B \to \neg A)) \tag{13}$$

To get to something like this from (12), rewrite 'A' there as 'B' and prefix in the new A's as required, getting close to (13), except that the final consequent now reads $A \to \neg B$ rather than, as desired, $B \to \neg A$: a loose end we leave the reader to tidy up.

We have given an even-handed treatment here to contrariety and subcontrariety, with not only κ and $\tilde{\kappa}$, but also σ and $\tilde{\sigma}$. But what becomes of the fact that we can define intuitionistic ¬ using κ and → (as (Def.1~), where ¬ and κ appear respectively as ~ and ¬, when it comes to the σ side of the picture? Here the role formerly played by → must be played by $\to_d$, which is why we gave the appropriate rules for the latter connective in (7), and what results is a definition of $\neg_d$:

(Def.1$\neg_d$) $\neg_d A = (\sigma A \to_d A)$

To check that this works, we need its right hand side to serve as a strongest subcontrary of A, given that σA is a subcontrary of A. First we show ('strongestness') that if B is a subcontrary of A, then B is implied by $\sigma A \to_d A$. This is a matter of the downward application of the rule $\to_d$ rule (7), which applies to the premiss $\sigma A \vdash A \vee B$ ('thinning in' σA to serve as the C in this application) to yield the desired conclusion $\sigma A \to_d A \vdash B$. Note that the choice of σA is immaterial here, since any other formula could have been introduced to weaken the subcontrariety assumption $\vdash A \vee B$. Next, we show, given that σA is a subcontrary of A, that $\sigma A \to_d A$ is a subcontrary of A. We may apply the $\to_d$ rule (7) upward taking C as σA and B as $\sigma A \to_d A$, to get the conclusion (14)

$$\sigma A \vdash A \vee (\sigma A \to_d A) \tag{14}$$

Since *ex hypothesi* we also have $\vdash A \vee \sigma A$, using intuitionistically acceptable principles for ∨, (15) follows

$$\vdash A \vee (\sigma A \to_{d} A) \tag{15}$$

according to which A and $\sigma A \to_{d} A$ are indeed subcontraries. Summarizing the situation for κ and σ in intuitionistic logic (extended therewith), then, using the Béziau-style definition of $\neg A$ as $A \to \kappa A$ on the one hand (appearing in different notation as (Def.1~) above, as in Section 2 of [9]) and the definition of $\neg_{d}A$ given by (Def.1$\neg_{d}$) as $\sigma A \to_{d} A$ on the other: $A \to \kappa A$ is not only a contrary, but the weakest contrary of A, while $\sigma A \to_{d} A$ is not only a subcontrary of A, but the strongest subcontrary of A, so that these compounds emerge as equivalent to $\neg A$ and $\neg_{d}A$ respectively.

If we take the background logic as classical logic rather than intuitionistic logic, this last distinction disappears. As noted in [9], Sections 1 and 2, this point lies at the heart of Béziau's example, that since classically any implication and its antecedent are subcontraries, $A \to \kappa A$ is not only, as in the intuitionistic case, a contrary of A, but a subcontrary of A. And since $\sigma A \to_{d} A$ implies any subcontrary of A, we have the forward direction of (16) for $\vdash\ = \ \vdash_{\mathrm{CL}}$ (augmented by the above rules governing all connectives appearing)

$$\sigma A \to_{d} A \dashv\vdash A \to \kappa A, \tag{16}$$

the backward direction of (16) being given by $(\kappa\sigma)_{\vdash}$, now taking $A \to \kappa A$ and $\sigma A \to_{d} A$ as the formulas there represented as κA and σA respectively. (This is a legitimate substitution because $(\kappa\sigma)_{\vdash}$ concerns an arbitrary contrary and an arbitrary subcontrary of A, and as we have seen $A \to \kappa A$ and $\sigma A \to_{d} A$ qualify respectively as a contrary and subcontrary of A.) Although this qualifies as a complete explanation of the collapse of our two negations in classical logic, the story would be lopsided if we left it here. The justification of the forward direction of (16) used the fact that $A \to \kappa A$ and A are subcontraries (according to $\vdash_{\mathrm{CL}}$), so considerations of duality lead us to expect an equally satisfactory justification, instead, in terms of the fact that $\sigma A \to_{d} A$ and A are (according to $\vdash_{\mathrm{CL}}$) contraries, to which we then append the observation that since $A \to \kappa A$ is the weakest contrary of A, it should follow from $\sigma A \to_{d} A$. The contrariety in question is obvious from the fact that, added to classical rules for the remaining connectives, the $\to_{d}$-rule (7) makes $C \to_{d} D$ equivalent to $C \wedge \neg D$,[9] but we prefer to conduct the discussion without appeal to properties of classical $\neg$ (thinking of the present primitives as prior to the addition of that connective). The point made above about the downward direction of (7)'s $\neg_{d}$-rule being strictly weaker – in the setting of intuitionistic logic – than a corresponding rule allowing side-formulas on the left applies also in the case of the $\to_{d}$-rule. This is no longer true in the case of classical logic, as we illustrate in the case of a single side-formula, D, say, in which case, we want to be able to pass, with the aid of the $\to_{d}$-rule as formulated, from $D, C \vdash A \vee B$ to $D, C \to_{d} A \vdash B$. We may do this with the aid of some implicational shunting of D to the right and back:

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[9] By contrast, in an intuitionistic setting, $C \to_{d} D$ is strictly weaker than $C \wedge \neg D$.

(*i*) $D, C \vdash A \vee B$	(given)
(*ii*) $C \vdash A \vee (D \to B)$	A classically (though not intuitionistically) correct transition[10]
(*iii*) $C \to_{d} A \vdash D \to B$	By the $\to_{d}$-rule
(*iv*) $D, C \to_{d} A \vdash B$	A transition from (iii) which is correct even intuitionistically.

Now put A for D and (i) becomes provable (i.e., holds for $\vdash\ =\ \vdash_{CL}$) and so delivers the corresponding special case of (iv), given here as (17):

$$A, C \to_{d} A \vdash B \tag{17}$$

This says that A and $C \to_{d} A$ are contraries, for any C, so in particular this holds in the case in which C is σA, as was to be shown. Having now rid ourselves of the apparent asymmetry involved in justifying the forward direction of (16) for $\vdash\ =\ \vdash_{CL}$ by starting from the subcontrariety of A and $A \to \kappa A$, we need to fix another apparent lopsidedness in the present account.

Our introductory paragraph recalled the point from Section 3 of [9] that instead of using (Def.1~), Béziau's point could equally well have been made using (Def.2~) there, which in the notation of the present note amounts to defining $\neg A$ as $A \to \kappa(A \to A)$, a move very much in the somewhat more familiar "define $\neg A$ as $A \to \bot$" tradition. In the course of developing intuitionistic logic with σ and κ, we offered (Def.1$\neg_{d}$), defining $\neg_{d}A$ to be $\sigma A \to_{d} A$, a treatment in the style of (Def.1~) in that we make a compound from A and a *local* subcontrary σA of A, local in the sense that σA has no pretensions of being a subcontrary of every formula. (This terminology was introduced in note 8 of [9].) Similarly, (Def.1~) in the present notation defines $\neg A$ by means of a compound (viz. $A \to \kappa A$) of A and a local contrary thereof. (Def.2~) likewise uses instead a compound of A and a *global contrary* thereof – i.e. a contrary not only of A but of every other formula too – whether $\kappa(A \to A)$ or $\bot$. For a fully rounded picture, we should then expect to see a "(Def.2$\neg_{d}$)" in terms of a *global subcontrary* – a subcontrary of every formula. In fact, we have already met with such a definition, stated immediately after (7), with $\neg_{d}A$ defined to be $\top \to_{d} A$. This leaves us with only the question of an analogue of the $A \to \kappa(A \to A)$ *definiens* for $\neg A$ in the case of $\neg_{d}A$. Can we construct, then, just using $\to_{d}$ and σ, a formula which is a global subcontrary in the same way that we could, using $\to$ and κ, construct a global contrary? The considerations of the latter half of note 8 in [9] show that just as a global contrary is an all-implying formula (essentially, the intuitionistic $\bot$), so a global subcontrary is nothing but a provable formula (essentially, $\top$): a consequence of the empty set by the current consequence relation.[11] The answer is that we can construct such a formula, in the shape of $\sigma(A \to_{d} A)$. For $A \to_{d} A$ itself is an all-implying formula: put A for C and apply the $\to_{d}$ rule (7) to obtain $A \to_{d} A \vdash B$. Thus taking B as $\sigma(A \to_{d} A)$, this together with the premiss $\sigma(A \to_{d} A) \vdash \sigma(A \to_{d} A)$ yields the conclusion $\vdash \sigma(A \to_{d} A)$ by $(\sigma)_{\vdash}$.

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[10] As is most easily seen in the framework of generalized consequence relations: take $\Vdash$ as $\Vdash_{CL}$ and rewrite (i) and (ii) as $D, C \Vdash A, B$ and $C \Vdash A, D \to B$; the latter then follows from the former by (9) from Section 1 of [9].

[11] The apparent failure of duality here – "provable" rather than "all-implied" to match "all-implying" is due to the built-in lopsidedness of the notion of a consequence relation. In terms of generalized consequence relations $\Vdash$, we have in the one case a formula B such that $B \Vdash \varnothing$ and in the other a B for which $\varnothing \Vdash B$, restoring symmetry.

This restoration of symmetry between the "σ and $\to_{\mathrm{d}}$" side of the story and the "κ and $\to$" side also points us to what, given the historical development of logical theory constitutes something of an asymmetry, one which goes some way to explaining the greater naturalness with which negation is approached from the κ and $\to$ direction than from the σ and $\to_{\mathrm{d}}$ direction. As we shall see, this point, when the underlying logic is strengthened from intuitionistic to classical, also explains why Béziau's example starts with contrariety and then adds (via (Def.1~), though the same point arises for (Def.2~)) subcontrariety rather than the other way around, thereby addressing a question raised in Section 2 of [9] (between (16) and (17) there): given that "classical negation can be expressed by one of its halves", why choose the half Béziau chooses rather than the other half? For the moment, we remain at the level of intuitionistic logic.

4. Apparent Left/Right Asymmetries Historically Explained

Consider the pure implicational fragment of IL, or the associated consequence relation $\vdash_{\mathrm{IL}}$. The constructive version of Béziau's point, reviewed in Section 3 of [9], considers what in the present notation would be the addition of a contrariety operator κ. and then proceeds to observe that we can boost this logic to the negation-implication fragment of IL in either of two ways: on the one hand we can subject κ to further logical demands so that it forms not just a contrary, but the weakest contrary, of a formula to which it attaches; on the other hand we can leave the logical powers of κ as they are and make a suitable definition (in either of two ways, given by (Def.1~) and (Def.2~)) so that the defined connective, rather than κ itself, has the logical powers of intuitionistic negation. (Whether we consider the availability of these two routes to the same destination to be paradoxical, as Section 5 of [9] suggests we should not, is immaterial in the present context.) The history of logic in the twentieth century began with the axiomatic method in the ascendant, the significance of the idea of a consequence relation emerging only later. For example, the first formalizations of classical and intuitionistic logic were axiomatizations. When a consequence relation is *atheorematic* in the sense that the empty set has no consequences, it will not be possible to single it out on the basis of an axiomatic presentation of a logic by the means used in setting up the Deduction Theorem (i.e., defining A to be a consequence of Γ when A can be reached from Γ and the axioms by means of – perhaps only certain of – the rules of the axiom system) since there will be no candidate axioms. The logic of dual intuitionistic implication is such an atheorematic logic. We can see this by recalling that this is a sublogic[12] of the logic of "dual classical implication", i.e., the consequence relation determined by the class of all valuations v such that $v(\mathrm{A} \to_{\mathrm{d}} \mathrm{B}) = \mathrm{T}$ iff $v(\mathrm{A}) = \mathrm{T}$ and $v(\mathrm{B}) = \mathrm{F}$, and the latter logic is atheorematic. (One sees by induction on the complexity of formulas that, denoting this consequence relation by $\vdash$, for any formula A whose leftmost variable is p_i, we have $\mathrm{A} \vdash p_i$, so for any v with $v(p_i) = \mathrm{F}$ we have $v(\mathrm{A}) = \mathrm{F}$, refuting the hypothesis that $\varnothing \vdash \mathrm{A}$.) Thus the starting point for the development dual to that reviewed above for $\to$, prior to the addition of κ, is simply not available for anyone in the sway of the axiomatic paradigm: there simply is no pure $\to_{\mathrm{d}}$ logic to extend with the addition of σ. Nor do matters fare any differently – as our last parenthesis shows – in the classical case.

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[12] Indeed, a proper sublogic: for example, in the classical but not in the intuitionistic case, we have $p \to_{\mathrm{d}} (p \to_{\mathrm{d}} q) \vdash q$.

The asymmetry just noted is due to the heritage of the axiomatic tradition, focussing attention even when consequence relations are considered, on the right rather than the left of the $\vdash$; a full symmetry is restored at its most elegant in the classical case when generalized consequence relations are considered, so that the $\to_{\mathrm{d}}$ and σ story can be told as the mirror image (reflected about the "$\Vdash$") of the $\to$ and κ story. Thinking instead entirely semantically, we can reproduce Figure 2 from Section 4 of [9] (whose own occurrence of $\neg$ would appear as κ in the present notation):

σA	$\to_{\mathrm{d}}$	A
– T	F	T
T F	T	F

Figure 1

The blank under σA now appears in the v(A) = T case, since the condition that A and σA are subcontraries on v tells us only that both cannot be false, whereas the blank in Figure 1 of [9] under κA (written there as $\neg$A) arose in the v(A) = F case, since this was the case in which contrariety relative to v tells us nothing about κA. But as with Figure 2 of [9], the blank has no import for the eventual truth-value of the formula in this case, since for σA $\to_{\mathrm{d}}$ A to be true we require that σA be true and A be false, and whether or not the former condition is met, the latter is not, so the whole formula ends up false.

There may seem to be a great artificiality about defining negation as in the classical case just reviewed, using, *inter alia*, the connective $\to_{\mathrm{d}}$ when C $\to_{\mathrm{d}}$ D already amounts to C $\wedge$ $\neg$D. Haven't we already got negation on our hands with the latter formula? Well, certainly we cannot define negation just in terms of $\to_{\mathrm{d}}$, as the atheorematic nature of the pure $\to_{\mathrm{d}}$ logic already shows, since $\neg(p \to_{\mathrm{d}} p)$ would be a theorem if $\neg$ were definable. (This point holds in the intuitionistic case also.) Another way of putting this is to say that it will do no good to object that we can simply reach for $\top$ or another other provable formula and plug this in for C in C $\to_{\mathrm{d}}$ D to get a formula equivalent to D, since, again, there simply are no such candidate formulas in the logic with only $\to_{\mathrm{d}}$. So the apparent artificiality reduces to that described above: a by-product of the axiomatic tradition which sees logic as primarily concerned with the articulation of the logical truths rather than principles of inference. This historically engendered asymmetry is what blocks a similar reaction to the case of defining $\neg$A as A $\to$ κA. One does not expect a reaction along the lines of "But if you have $\to$ then you already have negation since A $\to$ B amounts to $\neg$A $\vee$ B." (Recall we are considering the classical case. Though it is worth recalling that the pure implicational fragment of IL is sometimes called Positive Logic, notwithstanding the frequently useful classification of antecedents of "positively occurring" implicational subformulas as negative occurrences.) We know perfectly well that $\neg$ isn't definable in terms of $\to$ alone, even if it is when we help ourselves in addition to $\bot$. Here the presumed availability of an 'anti-theorem' $\bot$ is readily perceived as a novel addition to pure implicational logic, though the presumption of an available $\top$ is less likely to be noticed in the dual case because of a historically conditioned unfamiliarity with atheorematic logics – the axiomatic heritage, as we have called it.

5. Global and Local Contrariety

The last of the loose ends from [9] to be tied up concerns the relation between (Def.1~) and (Def.2~); both were tacitly in play in the previous paragraph, and we have seen that the parallel between them extends to the dual case of (Def.1$\neg_{\mathrm{d}}$) and (Def.2$\neg_{\mathrm{d}}$). (The present discussion addresses only this contrariety based case, and not the subcontrariety case.) Section 3 of [9] noted that Béziau could equally well, if somewhat less surprisingly, made his point by using a definition in the latter 'global contrary' style. In the current notation, given a contrary forming operator κ and a theorematic (i.e., non-atheorematic) consequence relation, we can always manufacture a $\perp$-like (i.e., all implying) formula by applying κ to a theorem. As remarked, this would be a global contrary: a contrary of every formula, chosen once and for all. Since we were assuming a suitable – i.e., at least intuitionistic in logical powers – implication connective, we took as a representative theorem a formula of the form $A \rightarrow A$. We also remarked that given $\rightarrow$, one had such a global contrary available if and only if one had local contraries. (See note 8 of [9].) But this leaves open an interesting question. If our envisaged 1-ary operator O, say, is assumed only to form a global contrary when attached to a provable formula, does it follow that O itself when attached to an arbitrary formula yields a local contrary, i.e., a contrary of that formula? We will write "κ" rather than "O" for the operator in question, not in order to prejudge this question with an affirmative answer but to make contact with the principles that emerged in Section 2 above. An affirmative answer would mean that whenever one could use a (Def.2~) definition, one could use a (Def.1~) style definition, with the same κ (or $\neg$ as it appears it in Sections 2 and 3 of [9]), and arrive at an equivalent *definiens*. This issue is essentially the same as an analogue for contrariety of the (Aσ) *vs.* (Rσ) contrast for subcontrariety with which we began. Instead of the axioms schema (Aκ) ($= A \rightarrow (\kappa A \rightarrow B)$) we may consider the following "Rule" version of a permuted form thereof, which we accordingly call (Rκ):

$$(\mathrm{R}\kappa)\qquad \frac{\vdash A}{\vdash \kappa A \rightarrow B}$$

(Alternatively, we could consider replacing the condition $(\kappa)_{\vdash}$ with a condition to the effect that $\kappa A \vdash B$ whenever $\vdash A$.) If κ satisfies (Rκ) then it has the property exploited in (Def.2~) of turning a provable formula into an all-implying (or *Falsum*) formula, so if (Aκ) followed from positive implicational logic + (Rκ), this would mean that we could always use the same operator to form local contraries, available therefore for service in (Def.1~).[13]

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[13] The least extension of positive logic closed under (Rκ) is close to a variant of Rasiowa's logic of semi-negation described in 2.7.9 of [11], the difference being that the latter requires also an extensionality principle (here stated in terms of consequence relations) to the effect that $A \rightarrow B, B \rightarrow A \vdash \kappa A \rightarrow \kappa B$. The original presentation at p.192 of Rasiowa (in [10]) uses a weaker underlying logic for $\rightarrow$.

The answer to the question just posed is, however, negative – even if the underlying $\to$ logic is classical, as we see from the following. Consider the language whose only connectives are $\to$ and κ, interpreted respectively as material implication and impossibility. That is, we have in mind (*Kripke*) *models* $\mathcal{M} = \langle W, R, V\rangle$ in which $V(p_i) \subseteq W$ for each propositional variable p_i, W being a non-empty set, with $R \subseteq W \times W$; the pair $\langle W, R\rangle$ is called a *frame*, and $\mathcal{M}$ a model on that frame. The truth of a formula A at $x \in W$ in such a model, written "$\mathcal{M} \models_x$ A" is defined inductively by:

$\mathcal{M} \models_x p_i$ if and only if $x \in V(p_i)$

$\mathcal{M} \models_x \mathrm{A} \to \mathrm{B}$ if and only if $\mathcal{M} \not\models_x$ A or $\mathcal{M} \models_x$ B

$\mathcal{M} \models_x \kappa\mathrm{A}$ if and only if for no y such that Rxy do we have $\mathcal{M} \models_y$ A

Say a formula is *valid* over a class of frames if it is true at every point in every model on any frame, and that a logic (as a set of formulas) is determined by a class of frames just in case its theorems are exactly the formulas valid over the class.[14] Consider the logic determined by class of frames $\langle W, R\rangle$ satisfying the condition that for all $x \in W$ there exists $y \in W$ with Rxy. (These are generally called serial frames in the modal logic literature, and the normal modal logic thereby determined is **KD**, so we are considering an implication-impossibility formulation of **KD**.) This logic is closed under the rule (Rκ) but it does not contain all instances of the schema (Aκ), showing the underivability of the latter from the former even in the presence of full classical implication. Indeed, adding (Aκ) to a complete axiomatization of this logic yields an axiomatization of the logic determined by the class of frames $\langle W, R\rangle$ in which R is reflexive – a formulation in the current primitives of the logic **KT** mentioned in Section 5 of [9]. Thus in (this version of) **KT**, classical negation is definable *à la* (Def.1~), though we write the defined connective as $\neg$ here, by: $\neg\mathrm{A} = \mathrm{A} \to \kappa\mathrm{A}$, as well as by means of the definition in the style of (Def.2~): $\neg\mathrm{A} = \mathrm{A} \to \kappa(\mathrm{A} \to \mathrm{A})$. On the other hand in the weaker logic originally defined – our version of **KD** – only the latter definition is available.

6. Semantical Postscript

Having introduced the Kripke semantics for modal logic for the purpose of that comparison between (Def.1~) and (Def.2~), we may as well conclude with a few remarks on the corresponding apparatus for intuitionistic logic with the various additional operators that have been in play in this discussion. Here we recall that in models $\mathcal{M} = \langle W, R, V\rangle$ we require that R is a partial ordering, in honour of which fact we shall instead it as $\leq$ (with $\geq$ for its converse), and that a persistence condition to the effect that for $x, y \in W$, if Rxy and $x \in V(p_i)$ then $y \in V(p_i)$. The inductive clauses in the definition of truth at a point in a model for $\wedge$, $\vee$, $\to$, we assume known, though that for $\neg$ is repeated here for the sake of comparison with that (due to Rauszer) for $\neg_{\mathrm{d}}$; though formulas of the form $\neg_{\mathrm{d}}\mathrm{A}$ are automatically persistent (in the sense that truth at $y \geq x$ follows from truth at x), we need to impose this as a special stipulation in the case of formulas of the forms $\kappa\mathrm{A}$, $\tilde{\kappa}\,\mathrm{A}$, $\sigma\mathrm{A}$, $\tilde{\sigma}\,\mathrm{A}$. We say further that for any $\mathcal{M} = \langle W, \leq, V\rangle$, $x \in W$:

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[14] If we want the logic as a consequence relation we take it to be the set of pairs $\langle \Gamma, \mathrm{A}\rangle$ such that at any point in a model on a frame in the class at which all formulas in Γ are true, A is true.

$\mathcal{M} \vDash_x \neg A$ if and only if for all $y \geq x$ $\mathcal{M} \nvDash_y A$

$\mathcal{M} \vDash_x \kappa A$ only if for all $y \geq x$ $\mathcal{M} \nvDash_y A$

$\mathcal{M} \vDash_x \tilde{\kappa} A$ if for all $y \geq x$ $\mathcal{M} \nvDash_y A$

$\mathcal{M} \vDash_x \neg_{\mathrm{d}} A$ if and only if for some $y \leq x$, $\mathcal{M} \nvDash_y A$[15]

$\mathcal{M} \vDash_x \tilde{\sigma} A$ only if for some $y \leq x$, $\mathcal{M} \nvDash_y A$

$\mathcal{M} \vDash_x \sigma A$ if for some $y \leq x$, $\mathcal{M} \nvDash_y A$

We have set out these clauses in so that the conditions for κ and $\tilde{\kappa}$ ($\tilde{\sigma}$ and σ) emerge as 'only if' and 'if' halves of the clause for $\neg$ (resp. $\neg_{\mathrm{d}}$), but given what has been said about persistence (together with the reflexivity of $\leq$), it can be easily checked that the above clauses for κ and σ are equivalent to the following more natural contrariety and subcontrariety conditions:

$\mathcal{M} \vDash_x \kappa A$ only if $\mathcal{M} \nvDash_x A$ and $\mathcal{M} \vDash_x \sigma A$ if $\mathcal{M} \nvDash_x A$

In spite of the unusual semantics with such "one way" rather than biconditional clauses governing the novel operators,[16] a standard canonical model argument provides a completeness proof for the extension of intuitionistic logic by the principles introduced in this note governing them, with respect to the class of all frames $\langle W, \leq \rangle$.[17]

References

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[2] Haskell B. Curry, *Foundations of Mathematical Logic*, McGraw-Hill, New York 1963. (Second Edn., Dover Publications, New York 1977.)

[3] Rajeev Goré, 'Dual Intuitionistic Logic Revisited', pp.252–267 in R. Dyckhoff (ed.), *Automated Reasoning with Analytic Tableaux and Related Methods*, Springer-Verlag, Berlin 2000.

[4] Lloyd Humberstone, 'Extensionality in Sentence Position', *Journal of Philosophical Logic* **15** (1986), 27–54; corrected *ibid.* **17** (1988), 221–223.

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[15] The corresponding clause for dual intuitionistic implication reads: $A \to_{\mathrm{d}} B$ is true at x just in case if and only if for some $y \leq x$, A is – while B is not – true at y.

[16] See also pp.419–421 of [5], for remarks on this phenomenon in model-theoretic semantics. Of course in purely valuational semantics, we already have an example of such uni-directional conditions in Béziau's treatment of negation (in [1]) by means of a condition on valuations requiring the negation of A true only if (as opposed to: if and only if) A is false (the "weakly boolean" valuations of [9], especially §2). See the references in [1] to the work of da Costa and others which inspired this feature.

[17] This material was an Appendix to [9] as originally delivered to the Australasian Association for Logic conference in Canberra, November–December 2002, and the acknowledgments made in the final note of [9] therefore apply here also, to which we add a word of thanks to a *Theoria* referee for suggesting some improvements.

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