

## Parts and Partitions

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### 0. Overview

Our object is to study the interaction between mereology and David Lewis' theory of subject-matters, elaborating his observation that not every subject matter is of the form: how things stand with such-and-such a part of the world. After an informal introduction to this point in Section 1, we turn to a formal treatment of the partial orderings arising in the two areas – the part-whole relation, on the one hand, and the relation of refinement amongst partitions of the set of worlds, on the other. (We follow Lewis – approximately – in identifying subject-matters with such partitions.) We emphasize a certain duality, formulated in (2.6) and (2.7) in Section 2, between the corresponding lattice operations – mereological joins with partition-lattice meets, mereological meets with partition-lattice joins. Section 3 presents some issues that are raised by consideration of the informally familiar idea of logical subtraction. These include, in particular, a problem about the need for a notion of independence different from the usual logical notion(s) going by that name. The apparatus of Section 2 promises to throw some light on this problem, as we indicate in Section 4. Section 5 ties up some loose ends and suggests an area in which further work would be desirable.

### 1. Parts vs. Aspects

Think of those pairs of cartoons—they appear in many newspapers, magazines, and puzzle compendia—in which the reader is challenged to list the ten (say) respects in which the two pictures, at first glance portraying the same scene, turn out on closer inspection to differ. The caption will read something along the lines of “Spot the Difference(s)” or “Spot the Flaws”, and when, it being too tedious to persist after finding most but not all of the promised discrepancies, you look up the solution, you read perhaps “(1) No smoke from the candle; (2) Extra flower in the vase; (3)...”, meaning that the picture on the right has omitted the smoke depicted on the left, and introduced an additional flower into the vase on the table at which a woman is sitting, and so on. Answer (2) here might well appear in a more explicit form: “6 flowers rather than 5 in the vase”; whether it does or not, let us imagine that that is the precise form of the flower-count discrepancy for a particular pair of pictures. One thing you are not supposed to list is that the number of flowers now fails to match the number of fingers on the woman's right hand whereas in the first picture – the picture on the left, that is – it did. It's not just that if that's the kind of difference you were to register, you would come up with far more than ten such differences. And it's not just that given the fact that there are six flowers adding such 'relational' differences would be a kind of double-counting. For the conventions of the genre dictate not only that you mustn't cite this difference *as well*, but that you may not cite it *instead*, either. The convention requires, roughly, that the differences to be registered are 'local' differences, localized either in

the sense of being intrinsic differences in what is depicted at corresponding regions of the scenes, or in the sense of being intrinsic differences in what is depicted for corresponding objects in the scenes. (We need to mention both cases since a depicted space-occupant might have ‘moved’ from one position in the one scene, to another in the other.) The ‘roughly’ is here because there may be no convenient way of registering as a local difference some such difference as: the woman is closer to the table in the second picture than she is in the first; in that case, the description just given may be the expected answer.

The kind of (for purposes of spot-the-differences puzzles) unwanted respect of difference mentioned above—difference over whether the number of flowers is the number of the fingers—is not the only kind of unintended response. Another would consist in mentioning, rather than the single difference of the added flower, the two differences, one of which is an additional flower-stem, and the other an additional flower-head. This would seem to reflect Rosch-type phenomena of natural classification: a flower is naturally classified, as a matter of psychological fact, as just that, rather than as a flower head with a stem attached to it.<sup>1</sup> But is this another kind of unintended response? The alternative would be to see the earlier example in the same light: as a case of unnatural – or at least unanticipated – division of the scene into objects (depicted). Just as we are not expected to subdivide the flower, so we are not expected to lump together the hand and the vase-contents, and form a scattered object about which to observe that, as represented in the first picture, for some number  $n$ , its hand-part has  $n$  fingers and its vase-part contains  $n$  flowers, while in the second picture this is not so. We can, after all, always redescribe extrinsic properties of objects in terms of intrinsic properties of larger objects with the original objects as parts.<sup>2</sup> But given a specification of which objects are under consideration, one not permitting arbitrary mereological aggregation or subdivision, the idea of localized information, information about the intrinsic properties of the objects provided by the specification, would seem to play at least a large role in how we respond to tasks like that of spotting the differences between two depictions of those objects differently disposed. Further investigation of that particular task – hardly a central cultural pursuit – will not concern us. Rather, we take up this idea that not every aspect of a scene is to be construed as a matter of how things stand with some participant in the scene, or with what is going on in a part of the scene, by substituting for the contrast: ‘aspect of a scene’/ ‘part of a scene’, the contrast: ‘aspect of reality’/‘part of reality’. We shall, in other words, play

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<sup>1</sup> On p.3 of Eleanor Rosch *et al.*, *Basic Objects in Natural Categories* (Working Paper No. 43, Language Behavior Research Laboratory, University of California, Berkeley, 1975), one reads that “the aim of the present research was to show that the world does, in a sense, contain ‘intrinsically separate things’.” The authors oppose a view which they say (p.2) “would be reasonable only if the world were entirely unstructured”. The notion of structure employed turns out to be psychological (defined in terms of stimulus discrimination), explaining the parenthetical ‘in a sense’. The research reported also bears more on the question (as philosophers might put it) of the type a given token is most readily classified as falling under, rather than—what is more to the point here—how a part of reality comes to attention as a token about whose type-status such a question might arise.

<sup>2</sup> See the third (new) paragraph on p.62 of David Lewis, *On the Plurality of Worlds* (Blackwell, Oxford 1986), for an illustration of this point. The intrinsic/extrinsic distinction amongst properties in play in our discussion is the same as that explained by Lewis; although the property of having parts related in such-and-such ways is intrinsic on this understanding, it would not be appropriate to regard it as non-relational, at least as picked out under the (schematic) description just given, with its conspicuous appeal to how the parts are related. A more detailed exposition of this and related points may be found in my ‘Intrinsic/Extrinsic’ (*Synthese* **108** (1996), 205–267).

“Spot the Differences” with possible worlds in place of cartoon pictures. In Section 4, the idea of ‘part-based’ or (more generally) localized respects of sameness and difference between worlds—informally introduced here and further elaborated in Section 2—will be used to throw some light on a thorny problem arising in the theory of logical subtraction, as explained in Section 3. Some residual issues are taken up in Section 5.

1988 was a busy year for the theory of aboutness. John Perry published a paper<sup>3</sup> claiming that the possible worlds framework was incapable of supporting a non-trivial account of a given statement’s being about a given subject matter—non-trivial in the sense of not making every statement turn out to be about every subject matter—and David Lewis published two papers articulating and applying just such an account in that allegedly inhospitable framework.<sup>4</sup> According to this account, a subject matter is an equivalence relation – or, if preferred, the associated partition – on the set of possible worlds, and a statement is (entirely) about a given subject matter when the statement has the same truth-value in each of any two worlds standing in that relation (or: lying in the same block of the partition). For our purposes, the important thing is that some but not all subject matters are what we might call ‘part-based’, in the sense that the relevant equivalence relation is exact similarity (‘duplication’) in respect of some part of the world(s). In ‘Statements Partly About Observation’, Lewis gives as examples of such subject matters: the 17th Century, where the equivalence relation holds between worlds whose 17th Centuries are exact duplicates – or between worlds neither of which has a (counterpart of our) 17th Century; the 1680’s, similarly understood; and styrofoam, a statement being (entirely) about this subject matter if and only if “whenever all the scattered styrofoam of one world is a duplicate of all the scattered styrofoam in the other (or neither world contains any styrofoam), then both worlds give the statement the same truth-value.”<sup>5</sup> But, he continues:

It is otherwise for other subject matters. For instance, consider the subject matter: how many stars there are. Two possible worlds are exactly alike with respect to this subject matter iff they have equally many stars. A statement is entirely about how many stars there are iff, whenever two worlds have equally many stars, the statement has the same truth-value at both. Maybe an ingenious ontologist could devise a theory saying that each world has its *nos-part*, as we may call it, such that the nos-parts of two worlds are exact duplicates iff those worlds have equally many stars. Maybe—and maybe not. We shouldn’t rely on it.

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<sup>3</sup> ‘Possible Worlds and Subject Matter’, pp.124–137 in Sture Allén (ed.), *Possible Worlds in Humanities, Arts and Sciences*, de Gruyter, Berlin 1989. (The copyright date is 1988; the conference whose proceedings are here collected was held in 1986.) Perry draws attention to the distinction with which we shall be concerned here, between parts of the world and aspects of the world, as a distinction between ‘chunks’ and aspects (using ‘part’ to subsume both), at the base of p.101 of his paper ‘From Worlds to Situations’, *Journal of Philosophical Logic* **15** (1986), 83–107, the general moral of which is supposed to be that the apparatus of possible worlds is too indiscriminating to draw many important distinctions.

<sup>4</sup> ‘Statements Partly About Observation’, *Philosophical Papers* **17**, 1–31, and ‘Relevant Implication’, *Theoria* **54**, 161–174. The final sentence of note 20 of the former paper applies no less to Perry’s argument (*op. cit.*, p.129) as to the earlier ruminations of Nelson Goodman which provoked it.

<sup>5</sup> ‘Statements Partly About Observation’, p.12; the inset quotation appearing next is from the same page. *Note for younger readers*: with the reference to styrofoam, Lewis is making a gently humorous allusion to an influential 1970 paper on mass terms (by Terence Parsons) which featured memorable examples along the lines of “All blue styrofoam is styrofoam”.

Rather, we should say that being exactly alike with respect to a subject matter may or may not be a matter of duplication between the parts of the worlds which that subject matter picks out.

One potential source of confusion needs to be cleared up straight away. The example of the number of stars may seem not to lend itself to a treatment in terms of parts of the world because it is the whole of a world, and not any proper part of it, that determines how many stars there are *altogether* in that world. This is a confusion because ‘part’ in this discussion does not mean ‘proper part’, so this opposition between whole and part is beside the point. The example could in any case just as easily have been given in terms of a statement’s being entirely about how many stars there are in the Milky Way, for instance. By analogy with what Lewis says about the 17th Century, worlds in which there is no (counterpart of the) Milky Way are equivalent *modulo* the subject matter in question, as are worlds in which the Milky Way does have counterparts and those counterparts are duplicates. Although the Milky Way is a part of the world, and the statement that there are so-and-so many stars in the Milky Way is a statement entirely about the Milky Way, there is no part of the world such that the subject matter *how many stars there are in the Milky Way* is the part-based subject matter determined by the Milky Way as part. For two worlds could be alike in respect of how many stars there are in the Milky Way without their Milky Way parts’ being duplicates. What would be true of ‘Milky-Way-nos parts’ is that (adapting what Lewis says in the passage quoted) the Milky-Way-nos parts of two worlds are exact duplicates if and only if those worlds have equally many stars in the Milky Way. And the ‘if’ part of this condition fails if we take the Milky-Way-nos-part to be the Milky Way. (Of course the ‘only if’ part will then hold, which is another way of saying that any statement as to how many stars there are in the Milky Way is indeed entirely about the Milky Way.) Note that I am introducing a little terminology of my own here. I am calling the subject matter we might label ‘how things are with respect to such-and-such a part of the world’, the subject matter determined by that part, and I am calling any subject matter thus determined by some part of the world a part-based subject matter. Like Lewis, I am here happy to say that a statement is about a part, meaning that it is about the subject matter determined by that part. We will be more fussy on this score in the following section, where we need to distinguish talk of parts from talk of the subject-matters determined thereby in order to describe the relations between the two domains.

Let me relate this a little to our opening discussion. The statement that in the second picture, there are six flowers in the vase, and the contrasting statement that in the first picture there are only five, is entirely about (the subject matter determined by) a part of the scene depicted—the same part in both cases—even though there is no part of the scene which determines the subject matter: how many flowers there are in the vase, or the subject matter: whether there are five, or instead six, flowers in the vase. There is also a part of the scene which the statement that there are, in the first picture, the same number of flowers as fingers, and the statement that in the second picture there are not the same number, is entirely about. The smallest such part is the fusion of the woman’s-right-hand part of the scene (the whole scene, that is) and the in-the-vase part of the scene – though again the question of which of these statements is true is not itself a part-based subject matter, and in particular it is not the subject matter determined by the part just mentioned.

## 2. Apparatus and Observations

A series of definitions and observations pertaining to the concepts informally introduced in the preceding section will lead conveniently to as much as we need to know (in fact, slightly more) for the application to the problem described in the following section. We follow Lewis in using capital letters from the middle of the alphabet as our default symbols for subject matters, though depart from him in thinking of these as denoting, specifically, the subject matter construed as partition (of the set of worlds) rather than the equivalence relation; if  $M$  is such a subject matter, then the associated equivalence relation will be denoted by  $\equiv_M$ . The partial ordering  $\leq$  (“is a refinement of”) of subject matters is understood in the usual way— for partitions—in terms of inclusion of equivalence relations:

$$(2.1) \quad M \leq N \Leftrightarrow \equiv_M \subseteq \equiv_N$$

As Lewis remarks, this necessitates some caution when it comes to talking of parts: while, considered as a part of the world the 1680’s constitute a part of the 17th Century, the subject matter determined by the latter bears the  $\leq$  relation to that determined by the former. We will use lower case letters from the middle of the alphabet to stand for parts of the world, or, as Lewis puts it,<sup>6</sup> parts of the world ‘in intension’, since we are really thinking of a part  $m$  here as something that assigns to each world (for which it is defined) the ‘ $m$ -part of that world’ (a part ‘in extension’). Unlike Lewis, in the interests of simplicity, we take the ‘something’ here to be a total function rather than a partial function, and further, take  $m(w)$  as given outright rather than defined in terms of counterparts. We use ‘ $\gg$ ’ to stand for the relation of duplication, and when  $m(v) \gg m(w)$ , for worlds  $v, w$ , we frequently abbreviate further to:  $v \gg_m w$ . Thus  $M$  is the subject matter determined by the part  $m$  just in case the relation  $\equiv_M$  is the relation  $\gg_m$ . In this case, let us say that  $\sigma(m) = M$ . In other words,

$$(2.2) \quad \text{For all } v, w: v \equiv_{\sigma(m)} w \Leftrightarrow v \gg_m w.$$

(Here and below, variables  $u, v, w$  range over some set  $W$ , fixed from the context as comprising all possible worlds under consideration.) For the part-whole relations on parts ‘in extension’ we use the notation  $\sqsubseteq$ , transferred across to parts in intension by the definition

$$(2.3) \quad m \sqsubseteq n \Leftrightarrow \text{for all } w, m(w) \sqsubseteq n(w)$$

Then the point illustrated (following Lewis) with the example of the 1680’s and the 17th Century above, is given in (2.4); in fact it is spatial rather than temporal parts that we mainly have in mind in the present discussion.

$$(2.4) \quad \sigma(n) \leq \sigma(m) \Leftrightarrow m \sqsubseteq n$$

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<sup>6</sup> On each of the first two pages of ‘Relevant Implication’.

The  $\Leftarrow$ -direction of (2.4) can be written as: if  $m \sqsubseteq n$  and  $v \gg_n w$ , then  $v \gg_m w$  (for all  $v, w \in W$ ); one can only duplicate the whole by duplicating all its parts, in other words. The  $\Rightarrow$ -direction will be touched on below.

We follow the example of the ‘ $\sqsubseteq$ ’ notation above and use ‘squared off’ versions of the familiar symbols for set-theoretic relations and operations for the corresponding mereological notions; thus ‘ $\sqcup$ ’ and ‘ $\sqcap$ ’ form the fusions and mereological intersections of two parts (‘in extension’, and thus ‘in intension’ by an understanding in the manner of (2.3)<sup>7</sup>). I say ‘intersection’ rather than ‘overlap’ because it is convenient to have the  $\sqcap$ -notation defined for arbitrary pairs of parts even those which do not overlap in the sense of having some non-empty part in common. Following our ‘squaring off’ convention, we denote this ‘empty part’ by:  $\square$ . At this point, more precision is called for in respect of the notion of spatial parts. Sometimes one has in mind the part-whole relation between space occupants, whether material objects (the cat’s head being a part of the cat) or quantities of matter (the blue styrofoam being part of the totality of styrofoam), and sometimes one has in mind the part-whole relation between spatial regions themselves. It is the second of these relations that we have in mind when thinking of the part-whole relation in spatial terms (as we shall be except for a few remarks in Section 5). This clarification may assuage the traditional hostility of mereologists to admitting a least element with respect to the  $\sqsubseteq$ -ordering.<sup>8</sup> It may seem excessive to postulate the existence of a ‘null individual’ as a part of every space-occupant; I hope it seems less so to allow ourselves to talk of the empty region. (If you prefer to think of regions as sets of spatial points, so that  $\sqsubseteq, \sqcup$  and  $\sqcap$  are just the familiar  $\subseteq, \cup$ , and  $\cap$ , then  $\square$  is just  $\emptyset$ .) For a given part-in-intension  $m$ , then  $m(w)$  is the  $m$ -region of world  $w$ ; and it should be emphasized that for  $v \neq w$ , we take  $m(v)$  to be a duplicate of  $m(w)$  only when  $m(v)$  and  $m(w)$  are alike in respect of all their intrinsic features, including here their having space occupants of such-and-such intrinsic characters at such-and-such locations within them. Less controversial than the least element  $\square$  of our lattice of parts, is the greatest element, to be denoted by  $\blacksquare$ ; whereas  $\square(w)$  is the empty region of  $w$ ,  $\blacksquare(w)$  is the fusion of all parts of  $w$ ; in fact for our purposes we may reasonably stipulate that  $\blacksquare(w) = w$  (for all  $w \in W$ ).

Since we shall only consider ‘static’ worlds, setting to one side all worries about time and change, the distinction between space-occupants and (at least) occupied spatial regions may seem not to make much of a difference beyond that, just exploited, of softening any opposition to  $\square$  (or more accurately to  $\square(w)$  for the various  $w \in W$ ), since this is a difference which makes itself felt especially when space-occupants are envisaged to move from one place to another. The places (spatial regions) are the things which we don’t think of as moving when this is what we are envisaging. Actually, however, the specifically ‘subregional’ understanding of the part-whole relation does have further repercussions, because of the modal analogue of this point. It’s not just that regions don’t take on new locations—since they *are* the locations—with the passage of time: a region doesn’t take on different locations at different worlds either. This allows us to assert

<sup>7</sup> That is,  $m \sqcup n$  is that function defined on  $W$  by the stipulation that for all  $w \in W$ :  $m \sqcup n(w) = m(w) \sqcup n(w)$ , and similarly for  $\sqcap$  (as well as for the operation of complementation introduced below.)

<sup>8</sup> See note 5 (p.13) of Peter Simons, *Parts: A Study in Ontology*, Clarendon Press, Oxford 1987.

$$(2.5) \equiv_{\sigma(m)} \cap \equiv_{\sigma(n)} = \equiv_{\sigma(m \sqcup n)}$$

For suppose that  $v \equiv_{\sigma(m)} \cap \equiv_{\sigma(n)} w$ , that is (by (2.2)), that  $m(v) \gg m(w)$  and  $n(v) \gg n(w)$ . From this it follows that  $m \sqcup n(v) \gg m \sqcup n(w)$  only because we are here dealing with spatial regions – rather than space occupants – of the worlds  $v$  and  $w$ . If instead we thought of  $m$  and  $n$  as assigning to any world respectively its styrofoam part and its vinyl part, then that would not follow: duplication of the parts would not suffice for duplication of their fusion because the parts could be differently related spatially.<sup>9</sup> (The converse implication for (2.5) is just a consequence of the fact that to duplicate a whole one must duplicate the parts, and this point is not sensitive to the “occupants”-vs.-“regions” decision as to what the relata are.)

We can write (2.5) less clumsily if we introduce some notation for the meet and join in the lattice of subject matters (the lattice of all partitions of  $W$ , that is<sup>10</sup>); let us write  $M \times N$  for the meet of subject-matters  $M, N$  (their greatest lower bound w.r.t. the ordering  $\leq$ ) and  $M + N$  for their join (least upper bound). Recall that  $\equiv_{M \cdot N}$  is  $\equiv_M \cap \equiv_N$ , whereas  $\equiv_{M+N}$  is the smallest equivalence relation (on  $W$ ) extending  $\equiv_M \cup \equiv_N$ , the extension typically being proper. Then (2.5) can be expressed as

$$(2.6) \sigma(m) \times \sigma(n) = \sigma(m \sqcup n)$$

What about the dual statement, (2.7) here?

$$(2.7) \sigma(m) + \sigma(n) = \sigma(m \sqcap n)$$

The  $\leq$ -half of (2.7) is readily established: since  $m \sqcap n \sqsubseteq m$ , by the ‘if’-direction of (2.4), we have  $\sigma(m) \leq \sigma(m \sqcap n)$ ; similarly case with  $\sigma(n)$  on the left; so  $\sigma(m) + \sigma(n) \leq \sigma(m \sqcap n)$ . The first and last steps of this argument involve just basic lattice-theoretic reasoning, about meets in the lattice of parts and joins in the lattice of partitions, respectively.

The  $\geq$ -half of (2.7) calls for a little preparation. In the first place, we need to be a bit more explicit about the algebra of parts, and in particular, to announce that this is to be taken as a boolean algebra; alongside our meets ( $\sqcap$ ) and joins ( $\sqcup$ ), assumed to satisfy the distributive law, that is, we have a complementation operation ( $\prime$ ) such that for any element  $m$  of this algebra,  $m \sqcap m' = \square$  and  $m \sqcup m' = \blacksquare$ . That is a formulation for the algebra of parts in intension. As for the extensional analogue, corresponding to each  $w \in W$ , we have a boolean algebra consisting of the regions of  $w$ , with  $(m(w))'$ , *alias*  $m'(w)$ , being all of  $w$  with the exclusion of its  $m$ -region.<sup>11</sup> If we say regions are disjoint when

<sup>9</sup> For instance, the styrofoam of  $v$  and of  $w$  could be located in a neat little blob of the same size in both, and likewise for the vinyl of  $v$  and  $w$ : but in  $v$  the blobs are closer together than they are in  $w$ , so the fusions differ intrinsically in respect of how extensive they are.

<sup>10</sup> Lewis, in the papers cited in note 4, does not regard all partitions as equally good candidates as subject matters, thinking of some as so “unnatural” as not to deserve the name; we make no such discriminations here.

<sup>11</sup> Though he does not think this gives a good model of the part-whole relation amongst material objects, Simons (in the opening paragraph of Section 3.5, p.127, *op. cit.*) accepts what he calls extensional mereology for regions of space, meaning that the part-whole relation here has the structure of a boolean algebra with its zero element

their meet is  $\square(w)$ , then this ‘mereological’ complement of a region (in  $w$ ) is simply the largest region disjoint from the given region. Next, we need to recall the concept of orthogonality from Lewis’s ‘Relevant Implication’: subject matters  $M$  and  $N$  are *orthogonal* when for all  $u, w \in W$ , there exists  $v \in W$  such that  $v \equiv_M u$  and  $v \equiv_N w$ . (As Lewis remarks, the orthogonality of the partitions  $M$  and  $N$  amounts to saying that every block of  $M$  overlaps every block of  $N$ .<sup>12</sup>) Using this concept, finally, we can state a version of the Principle of Recombination from Lewis’ *On the Plurality of Worlds*, to which (pp.87–92) the reader is referred for justification (as well as certain qualifications not to the point here), in the following form: disjoint parts determine orthogonal subject-matters. Without further ado, we accept/impose this condition, which we formulate as:

(Recomb.) *If  $m \sqcap n = \square$  then  $\sigma(m)$  and  $\sigma(n)$  are orthogonal.*

Since  $\equiv_{\sigma(m)}$  and  $\equiv_{\sigma(n)}$  are the relations of having the respective parts  $m$  and  $n$  duplicated (see (2.2) above), this amounts to saying that when these are disjoint regions, given any two worlds there is a third which duplicates the one in respect of the one region and the other in respect of the other. (Below, we shall derive the converse of (Recomb.) from another principle to be introduced presently.)

We are now in a position to address the  $\geq$ -direction of (2.7) above. We must show that  $\sigma(m \sqcap n) \leq \sigma(m) + \sigma(n)$ . Suppose, then, that worlds  $u$  and  $w$  lie in the same block of  $\sigma(m \sqcap n)$ , which means that their  $(m \sqcap n)$ -parts are duplicates. We need to show that  $u$  bears the relation  $\equiv_{\sigma(m) + \sigma(n)}$  to  $w$ , and this is the join (in the lattice of all equivalence relations on  $W$ ) of  $\equiv_{\sigma(m)}$  with  $\equiv_{\sigma(n)}$ . Now quite generally, a necessary and sufficient condition for the join of two equivalence relations to hold between two objects  $x$  and  $z$  is that there should exist for some  $k \geq 0$ , objects  $y_1, \dots, y_k$  such that each  $y_i$  bears one or other of those relations to  $y_{i+1}$ , where  $x = y_1$  and  $z = y_k$ .<sup>13</sup> In particular, then, a sufficient condition obtains when  $k = 1$ . Our  $x$  and  $z$  are the worlds  $u$  and  $w$ , so it will suffice to show that there exists a  $v$  with  $u \gg_m v \gg_n w$ . Recall that we are given that  $u \gg_{m \sqcap n} w$ . Now consider the region  $m \sqcap n'$ . Since it is disjoint from  $n$ , (Recomb.) tells us that there is a  $v$  such that  $u \gg_{m \sqcap n'} v \gg_n w$ . Now this  $v$  will turn out to be the  $v$  promised, for which  $u \gg_m v \gg_n w$ . For, since  $v \gg_n w$  and  $m \sqcap n \sqsubseteq n$ ,  $v \gg_{m \sqcap n} w$ ; but  $u \gg_{m \sqcap n} w$ , so  $u \gg_{m \sqcap n} v$ . Since also  $u \gg_{m \sqcap n'} v$ , we have  $u \gg_{(m \sqcap n) \sqcup (m \sqcap n')} v$ , which is to say:  $u \gg_m v$ . As we already have  $v \gg_n w$ , we are done.

The principles (2.6) and (2.7) together mean that the mapping we have denoted by  $\sigma$  is a dual homomorphism from the (lattice reduct of the) boolean algebra of parts into the

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removed. (Of course, we have taken the further liberty, in the hope that the passage from material objects to spatial regions makes it more palatable, of leaving the zero element in place.)

<sup>12</sup> This gloss makes clear that the expected irreflexivity of the relation of orthogonality is not quite forthcoming, since the one-block partition is orthogonal to itself. Since Lewis excludes the one-block partition as a genuine subject matter, this anomaly—to which my attention was drawn by Daniel Nolan—does not arise for him.

<sup>13</sup> See for example Theorem 4.6 on p.15 of S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York 1981.

lattice of subject matters.<sup>14</sup> If we want to bring the top and bottom elements into the picture, we can baptize the coarsest and finest partitions of  $W$  as  $\mathbf{1}$  and  $\mathbf{0}$  respectively. (Thus  $\equiv_1$  and  $\equiv_0$  are the universal relation and the identity relation on  $W$ , *alias*  $\gg_{\square}$  and  $\gg_{\blacksquare}$ ) Then we can fill out the story with

$$(2.8) \quad \sigma(\blacksquare) = \mathbf{0} \qquad \qquad \qquad \sigma(\square) = \mathbf{1}$$

In fact  $\sigma$  is also injective, since if  $\sigma(m) = \sigma(n)$  then, as  $\sigma(m) \leq \sigma(n)$  and  $\sigma(n) \leq \sigma(m)$  in that case, so by (2.4),  $n \sqsubseteq m$  and  $m \sqsubseteq n$  and therefore  $m = n$ . Thus inside the lattice of partitions of  $W$ , there sits an inverted copy of the lattice of parts (in intension). We give a miniature illustration of this in Figure 1 below, after which we continue to put (2.7) to further work. But first, we must comment on the role of (2.4) in the argument just given.

The plausibility of the  $\Leftarrow$ -direction of (2.4) was argued above: to duplicate the whole, one must duplicate all the parts. It is, however, the  $\Rightarrow$ -direction of (2.4) which is actually appealed to in the argument just given. In an axiomatic presentation of this material, it may seem preferable to derive this from a more fundamental piece of modal metaphysics than to postulate it outright, much as we appealed to (Recomb.) to justify (part of) (2.7) above. The principle in the background here would be to the effect that there are at least two (intrinsic) ways for any non-empty part of the world to be:

(Variety) *For all  $m \neq \square$ , there exist  $v, w \in W$  such that not  $m(v) \gg m(w)$ .*

With the help of this principle, together with (Recomb.), we can justify the  $\Rightarrow$ -direction of (2.4). Given  $m$  and  $n$  for which it is not the case that  $m \sqsubseteq n$  we must find  $v, w \in W$  with  $v \gg_n w$  but not  $v \gg_m w$ . Since under the hypothesis given  $m \sqcap n' \neq \square$ , (Variety) tells us that there are  $v, u \in W$  with not  $m \sqcap n'(v) \gg m \sqcap n'(u)$ . Thus by (Recomb.) there exists  $w \in W$  with  $w \gg_{m \sqcap n'} u$  and  $w \gg_n v$ ; this pair  $v, w$ , then satisfy our requirements that  $v \gg_n w$  and not  $v \gg_m w$ .

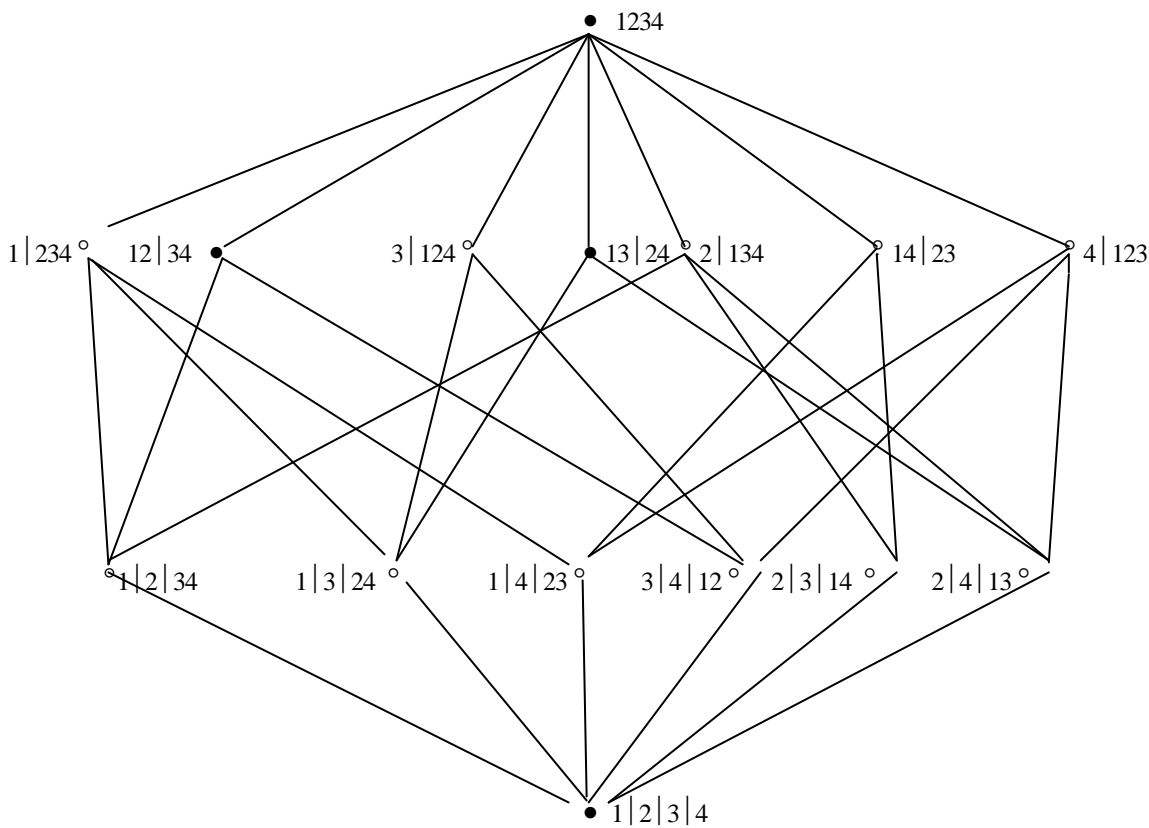
(Variety) itself also gives us the converse of (Recomb.), by the following argument. Suppose – for a contradiction – that  $\sigma(m)$  and  $\sigma(n)$  are orthogonal while  $m \sqcap n \neq \square$ . By (Variety), the latter assumption promises  $v, w \in W$  with non-duplicate  $(m \sqcap n)$ -parts. Since  $\sigma(m)$  and  $\sigma(n)$  are orthogonal, there exists  $u \in W$  with  $m(u) \gg m(v)$  and  $n(u) \gg n(w)$ . As  $(m \sqcap n) \sqsubseteq m$  and  $(m \sqcap n) \sqsubseteq n$ ,  $m \sqcap n(u)$  duplicates each of  $m \sqcap n(v)$  and  $m \sqcap n(w)$  (“to duplicate the whole one must duplicate the parts”), which is impossible since this would imply that  $m \sqcap n(v) \gg m \sqcap n(w)$  after all.

We proceed to the promised pictorial illustration of the boolean sublattice of parts as anti-isomorphically embedded (by  $\sigma$ ) in the lattice  $\Pi(W)$  of partitions of  $W$ . (We do not distinguish notationally between the partition lattice and its universe or ‘carrier’.) With increasing cardinality for  $W$ , the cardinality of  $\Pi(W)$  increases very rapidly, as we shall illustrate presently. For this reason we shall be concerned only with a ‘toy’ example, supposing that our set of parts (in intension)—call it  $\mathbf{M}$  (and again we do not distinguish

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<sup>14</sup> The lattice reduct here (a ‘boolean lattice’) is of course distributive, in striking contrast to the partition lattice; it is well known that no algebraic identity (such as the distributive law) holds in all partition lattices unless it holds in all lattices.

between this set and the boolean algebra with it as universe and operations  $\sqcap$ ,  $\sqcup$ , and  $\prime$ —is exhausted by two disjoint regions  $m$  and  $n$ , each of which can be in either of two qualitative states. (The letters ‘ $m$ ’ and ‘ $n$ ’, which generally function as variables ranging over parts, are used in discussion of this example, here and below, as constants.)  $\mathbf{M}$  then has four elements, namely these two and their fusion ( $= \blacksquare$ ) and intersection ( $= \square$ ). (Thus  $n = m'$ .) If we label the qualitative states for each of  $m$  and  $n$ , “on” and “off”, we can describe an element of  $W$  with an ordered pair of labels, giving  $m$ ’s state and  $n$ ’s state in that order; thus the four elements of  $W$  are described by  $\langle on,on \rangle$ ,  $\langle on,off \rangle$ ,  $\langle off,on \rangle$  and  $\langle off,off \rangle$ .<sup>15</sup> For brevity, we re-baptize these as ‘1’, ‘2’, ‘3’, and ‘4’ respectively. This gives a diagrammatically manageable size to  $\Pi(W)$ , which has the 15 elements depicted below. To reduce clutter, we use the vertical bar notation for partitions: all braces and commas are dropped, and distinct blocks are separated by bars. Thus, for example,  $\{\{1,2,3,4\}\}$  (the  $\mathbf{1}$  of the lattice) is denoted by “1234”, and  $\{\{1\}\{2,3,4\}\}$  by “1|234”:



**Figure 1**

To see how  $\sigma$  works here, let us begin with  $\sigma(m)$ , the subject matter determined by  $m$ . We have  $1 \equiv_{\sigma(m)} 2$  and  $3 \equiv_{\sigma(m)} 4$ , because 1 and 2 are worlds alike in having  $m$  in the ‘on’ state,

<sup>15</sup> Note that though for simplicity we have used the same labels for the  $m$ -position and the  $n$ -position, there is no assumption that the state labelled ‘on’ for  $m$  is qualitatively (intrinsically) like  $n$ ’s ‘on’ state – and likewise with ‘off’; nothing we do involves a comparison between the occupant of the first position of a pair and the occupant of the second position of that or any other pair.

while 3 and 4 agree in that  $m$  is ‘off’. Turning to  $n$ , what we find, for corresponding reasons, is that  $1 \equiv_{\sigma(n)} 3$  and  $2 \equiv_{\sigma(n)} 4$ . Thus  $\sigma(m)$  and  $\sigma(n)$  are respectively  $12|34$  and  $13|24$  (orthogonal partitions, as required by (Recomb.)). Their fusion and intersection are taken care of by (2.8) above:  $\sigma(m \sqcup n) = \sigma(\blacksquare) = 1|2|3|4$  and  $\sigma(m \sqcap n) = \sigma(\square) = 1234$ . In Figure 1, these four elements making up the image of  $\mathbf{M}$  under  $\sigma$  appear as solid rather than hollow circles (and of course the  $\leq$ -ordering given in the diagram reverses the boolean  $\sqsubseteq$ -ordering).

If we had started with three ‘basic’ regions instead of two, still allowing each to be in either of two qualitative states, we would have eight worlds in  $W$ ; this would have given us, instead of 15, a total of 4,140 partitions to deal with. Had we, sticking with the two initial regions  $m$  and  $n$ , allowed each of them to be in any one of three qualitative states, we should have had nine worlds and an alarming 21,147 partitions.<sup>16</sup> What I have referred to here as the set of basic regions is a special case of what we could call a mereological partition: a collection of (mereologically) pairwise disjoint elements of  $\mathbf{M}$  whose fusion is the top element (in terms of  $\sqsubseteq$ )  $\blacksquare$ . Such partitions are not of course to be confused with the (set-theoretical) partitions of  $W$  which constitute the subject matters. However, more is involved than just this, since our basic regions do not just partition  $M$  but are all *atoms* in the sense that nothing lies strictly  $\sqsubseteq$ -between them and the bottom element  $\square$ . There is no suggestion that for all plausible real-life choices of  $\mathbf{M}$ , this should be an atomic boolean algebra: it is simply something forced on us by the desire to have a tractably small (in particular, a *finite*) example.

We have stressed the role of (2.7) in (alongside (2.6)) revealing the subject-matter assignment map  $\sigma$  to be an order-inverting injective (*i.e.*, one-one) homomorphism from  $\mathbf{M}$  into  $\Pi(W)$ . It is worth illustrating its bearing on the ‘entirely about’ discussion from Section 1. One could justify directly, with the aid of (Recomb.), the following rather surprising looking principle:

(2.9) *If a statement  $S$  is entirely about  $\sigma(m)$  and also entirely about  $\sigma(n)$ , then  $S$  is entirely about  $\sigma(m \sqcap n)$ .*

The converse of (2.9) is obvious enough from Lewis’s discussion: since  $m \sqcap n$  is part of  $m$ , anything entirely about (the subject matter determined by) the former is *a fortiori* entirely about (“ $\sigma$  of”) the latter – *cf.* the case of the 1680’s and the 17th Century; and likewise of course in the case of  $m \sqcap n$  and  $n$ . What is less obvious is that the *only* way for  $S$  to be entirely about the two regions is by being about their intersection, as (2.9) says. As intimated above, one could establish this *ab initio* by appeal to (Recomb.): suppose that  $S$  is entirely about  $m$  and entirely about  $n$  but not entirely about  $m \sqcap n$ ; then there exist worlds  $u, v$ , which are  $(m \sqcap n)$ -duplicates but differ on the truth-value they assign to  $S$ . By (Recomb.), there is a world  $w$  which is an  $(m \sqcap n)$ -duplicate of  $u$  and an  $n$ -duplicate of  $v$ . Since  $u$  and  $v$  are  $(m \sqcap n)$ -duplicates, and  $w$  is an  $(m \sqcap n)$ -duplicate of  $u$ ,  $w$  is an  $m$ -duplicate of  $u$  (as  $m = (m \sqcap n) \sqcup (m \sqcap n')$ ). Now, with  $S$  being entirely about  $m$ ,

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<sup>16</sup> With  $r$  as the number of distinct regions and  $s$  as the number of distinct states allowed per region, we get  $s^r$  worlds; the number of partitions of a  $k$ -element set is given by what is called the  $k^{\text{th}}$  Bell number, whose values for  $k = 8, 9$ , as given in the text, I have taken from p.26 of Martin Gardner’s ‘Mathematical Games’ column on this topic in *Scientific American* **238** (Issue 5), May 1978, 24–30.

this implies that  $w$  and  $u$  have to assign the same truth-value to  $S$ , and since  $S$  is also entirely about  $n$ , the  $n$ -duplicates  $v$  and  $u$  must also agree on the truth-value of  $S$ . Since  $u$  and  $v$  assign different truth-values to  $S$ , we have a contradiction.

However, rather than argue from scratch as above, we can simply deduce (2.9) from (2.7), once we realize that to say a statement is entirely about a subject matter is to make a certain  $\leq$ -claim. As usual, we regard a proposition as a subset of  $W$ , and denote by  $\|S\|$  the proposition expressed by  $S$  (the set of worlds at which  $S$  is true). If  $S$  is contingent then the subject matter (cf. the quotation from Lewis in Section 1) whether-or-not- $S$ , we denote by  $\iota S?$ , defined as  $\{\|S\|, W \setminus \|S\|\}$ , where ‘ $\setminus$ ’ stands for relative complementation. (If  $S$  is non-contingent, this set will have the empty set as one of its elements and so not qualify as a partition, in which case  $\iota S?$  is obtained by removing  $\emptyset$ ; the result will simply be the partition  $\mathbf{1}$ .) Now to say that  $S$  is entirely about subject matter  $M$  is simply to say that lying in the same block of  $M$  implies agreement in respect of whether  $S$ ; so to say that  $S$  is entirely about  $M$  is to say that  $M \leq \iota S?$ . Under this reformulation (2.9) can be expressed as:

(2.10) *If  $\sigma(m) \leq \iota S?$  and  $\sigma(n) \leq \iota S?$ , then  $\sigma(m \sqcap n) \leq \iota S?$ .*

But as general lattice-theoretical point, if two elements are  $\leq$  an element then their join is  $\leq$  that element. So the antecedents of (2.10) trivially imply that  $\sigma(m) + \sigma(n) \leq \iota S?$ . Thus (2.10) itself follows once we know that  $\sigma(m \sqcap n) \leq \sigma(m) + \sigma(n)$ : but this is precisely the content of the  $\geq$ -half of (2.7).

The “ $M \leq \iota S?$ ” way of recording  $S$ ’s being entirely about  $M$  is helpful in allowing us to read off various other properties of aboutness from the mere use of the partial order notation “ $\leq$ ”. For example, from the fact that  $S$  is entirely about  $M$  and that  $N$  is a refinement of  $M$ , it follows that  $S$  is entirely about  $N$ ; with the present notation, all that is involved is transitivity:

(2.11) *If  $M \leq \iota S?$  and  $N \leq M$  then  $N \leq \iota S?$*

Of slightly greater interest is the following consequence of the fact that the lattice operations (of  $\mathbf{M}$ ) are monotonic with respect to the ordering  $\leq$ :

(2.12) *If  $M_1 \leq \iota S_1?$  and ... and  $M_k \leq \iota S_k?$  then  $M_1 \times \dots \times M_k \leq \iota S_1? \times \dots \times \iota S_k?$*

In other words: if the statements  $S_i$  are respectively about the subject matters  $M_i$ , then any worlds agreeing in respect of all those subject matters, agree as to whether  $S_1$  is true, and agree as to whether  $S_2$  is true, and so on. We can put this observation to use if we couple it with another simple consequence of the definitions given:

(2.13) *For any truth-functional compound  $\#(S_1, \dots, S_k)$  of the statements  $S_1, \dots, S_k$ :*  
 $\iota S_1? \times \dots \times \iota S_k? \leq \iota \#(S_1, \dots, S_k)?$

Putting (2.12) and (2.13) together, we have:

(2.14) For any truth-functional compound  $\#(S_1, \dots, S_k)$  of the statements  $S_1, \dots, S_k$ , if  $M_1 \leq \imath S_1?$  and ... and  $M_k \leq \imath S_k?$ , then  $M_1 \times \dots \times M_k \leq \imath \#(S_1, \dots, S_k)?$

A special case of (2.14) – the case arising when the  $M_i$  are all equal – is stressed by Lewis, in his discussion (in ‘Statements Partly About Observation’) of variations under the general heading ‘(Special) Composition Principle’: if each of  $S_1, \dots, S_k$ , is entirely about a given subject matter, then so is  $\#(S_1, \dots, S_k)$ .

In fact, this last special case is quite suggestive for logical purposes, because although the usual account of what it is for a statement  $S_{k+1}$  to be a consequence of statements  $S_1, \dots, S_k$ , would be in terms of the relationship

$$(*) \quad \|S_1\| \cap \dots \cap \|S_k\| \subseteq \|S_{k+1}\|$$

the following distinct relationship also satisfies all the formal conditions on a consequence relation

$$(**) \quad \imath S_1? \times \dots \times \imath S_k? \leq \imath S_{k+1}?$$

The two relationships have been distinguished under the names ‘inference’ and ‘supervenience’ (-determined consequence relation); our special case of (2.14) then says that a truth-functional compound of given components is a ‘supervenience’ style consequence of those components, even in those cases in which it is not a consequence in the familiar ‘inference’ related sense.<sup>17</sup>

Our final application of (2.7) will be in the service of a comment on a something Lewis says in ‘Relevant Implication’ in a section thereof entitled ‘Orthogonality and Connection’. In it, he remarks (indeed, *proves*) that when two subject matters overlap (as he puts it) they are connected. By ‘connected’ Lewis means ‘not orthogonal’. He explains overlapping (p.166) as having some subject matter as a common part, where the one-block partition we have been calling **1** is excluded from consideration as a candidate common part; while the terminology of overlapping may suggest the relation between  $M$  and  $N$  which holds when  $M \times N \neq \mathbf{0}$ , in fact Lewis is taking over (rather than dualizing) the vocabulary from the lattice of parts to the lattice of partitions, so what is actually meant, in our notation, is that  $M + N \neq \mathbf{1}$ . What he shows, then, is the following:

(2.15) If  $M$  and  $N$  are orthogonal, then  $M + N = \mathbf{1}$ .

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<sup>17</sup> A discussion of this distinction, and of the interrelations between the two notions of consequence, may be found in my papers ‘Some Structural and Logical Aspects of the Notion of Supervenience’, *Logique et Analyse* **35** (1992), 101–137, and ‘Functional Dependencies, Supervenience, and Consequence Relations’, *Journal of Logic, Language and Information* **2** (1993), 309–336; because the uninterpreted language of sentential logic is there to the fore, the role played by possible worlds in the (\*) and (\*\*) of the present text is played instead by suitable valuations (truth-value assignments).

As Lewis remarks, and illustrates with an example, the converse of (2.15) is false.<sup>18</sup> Indeed, we can see several counterexamples to the converse in Figure 1 above:  $1|234$  and  $12|34$  have the totally undiscriminating partition  $\mathbf{1}$  as their lattice sum (their most refined common coarsening), though they are not orthogonal; the same is true for  $1|234$  and  $2|3|14$ . Such counterexamples, Lewis calls cases of connection without overlap; after describing one, he adds (p.168):

But I find it hard to think of a natural example of connection without overlap. Maybe such cases should be excluded by a constraint on which equivalence relations count as genuine subject matters.

Now, whatever the merits of the general proposal toyed with here, we should at least pause to observe that the unwanted cases—counterexamples to the converse of (2.15)—do not arise amongst the part-based subject matters. For suppose that  $M$  and  $N$  are respectively  $\sigma(m)$  and  $\sigma(n)$  for  $m, n \in \mathbf{M}$ . Then we can argue:

$$\begin{aligned} \sigma(m) + \sigma(n) = \mathbf{1} &\Rightarrow \sigma(m \sqcap n) = \mathbf{1} \\ &\Rightarrow m \sqcap n = \square \\ &\Rightarrow \sigma(m), \sigma(n) \text{ orthogonal} \end{aligned}$$

where the steps are justified by (2.7), (2.8) together with the injectivity of  $\sigma$ , and (Recomb.), respectively.<sup>19</sup>

When Lewis writes of “such cases”—what I just called the unwanted cases—that perhaps they “should be excluded by a constraint on which equivalence relations count as genuine subject matters”, the cases to be excluded are not individual equivalence relations or (as in our preferred formulation) individual partitions, but *pairs* thereof, since the converse of (2.15), like (2.15) itself, is a condition on pairs  $M, N$ , of partitions, not a condition on partitions. In this respect the current proposal differs from Lewis’ occasional reluctance to classify as subject matters, partitions of  $W$  with too few or too many distinctions ( $\mathbf{1}$  and  $\mathbf{0}$ : “degenerate” cases), as well as those making distinctions which are insufficiently “natural” (cf. note 10 above). But although the cases to be excluded are thus relationally characterized, presumably the intention is to find a *monadic* condition on partitions, such that amongst all partitions meeting that condition, no pairs are to be found which will violate the converse of (2.15). And as we have just seen, being part-based is precisely such a condition.

Since being amongst the part-based subject matters is thus *sufficient* for two partitions to be “orthogonal-if-they-sum-to-1”, it is appropriate to raise the question of whether this condition is also *necessary*. That is, if we have a collection of partitions all pairs of which are orthogonal-if-they-sum-to-1, must every partition in the collection be part-based? Figure 1 again returns a negative answer. Aside from  $\mathbf{1}$  and  $\mathbf{0}$ , the part-based partitions, recall, are  $12|34$  and  $13|24$ , which are the subject matters we might call ‘Whether region  $m$  is *on* or *off*’ and ‘Whether region  $n$  is *on* or *off*’, respectively. Since

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<sup>18</sup> A simpler example along with some other pertinent information may be found in Appendix C of my paper ‘A Study in Philosophical Taxonomy’, *Philosophical Studies* **83** (1996), 121–169.

<sup>19</sup> Since we appealed to (Recomb.) in justifying the  $\geq$ -direction of (2.7), it is worth pointing out that it is the  $\leq$ -direction which is needed for the above argument.

these are the only states either region can assume, we might just as well describe the subject matters as: how things are with respect to region  $m$  (with region  $n$ ). Now compare  $14|23$ , orthogonal to each of those two (and summing to  $\mathbf{1}$  with each of them), which is not a part-based partition. Spelling out the label more explicitly, we have here the partition:

$$\langle on,on \rangle, \langle off, off \rangle \mid \langle on,off \rangle, \langle off,on \rangle.$$

This is precisely one of those *aspects* of reality which (as with the examples mentioned in Section 1) does not correspond to how things stand with respect to some *part* of reality: the subject matter is whether regions  $m$  and  $n$  agree in respect of being *on* or *off*.

### 3. The Notion of Independence in the Theory of Logical Subtraction

As indicated in our initial overview, we change the subject for the present section, introducing a problem on which we shall suggest, in Section 4, that the apparatus developed above promises to throw considerable light. The problem concerns logical subtraction, and so we must begin with some remarks as to what is meant by this phrase.

The idea of logical subtraction is the idea of a binary connective whose work is to ‘undo’ the work of conjunction; conjoining one statement with another makes for a statement which typically requires more for its truth than either alone, while subtracting one from another should yield a statement requiring less for its truth than if the subtraction were effected, the requirements deleted being those for the truth of the statement subtracted. Under conditions on statements  $A$ ,  $B$  (or formulas  $A$ ,  $B$ , if we are thinking of giving an extension of classical propositional logic accommodating the new connective) to be gone into in more detail presently, the thought is that  $(A \wedge B) - A$  should be equivalent to  $B$ .<sup>20</sup> The need for special conditions to be satisfied will be evident in a moment. Suppose that, whatever exactly those conditions might be, they are satisfied when  $A$  and  $B$  are distinct atomic sentences, as represented by propositional variables, or ‘sentence letters’, in the formal setting just parenthetically alluded to; we use letters from the range ‘ $p$ ’, ‘ $q$ ’,... in this capacity. A truth-functional treatment of the

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<sup>20</sup> An excellent introduction to the topic is provided by R. A. Jaeger, ‘Action and Subtraction’, *Philosophical Review* **82** (1973), 320-329, the title of which is inspired by a well known remark of Wittgenstein’s in the *Philosophical Investigations*, and the reply by J. L. Hudson, ‘Logical Subtraction’, *Analysis* **35** (1975), 130-135. Additional published discussions include R. A. Jaeger, ‘Logical Subtraction and the Analysis of Action’, *Analysis* **36** (1976), 141-146, and J. Hornsby, ‘Arm Raising and Arm Rising’, *Philosophy* **55** (1980), 73-84. I have heard the suggestion from several people (including Peter Lavers and David Makinson) that an approach to logical subtraction analogous to the treatment of contraction in the theory of belief revision may show promise, though I would prefer something more ‘objective’ than this (not relying on anything like relative entrenchment of beliefs, for example). My own first venture into this area was a paper ‘Reciprocal Connectives’ delivered in 1973 to the weekly philosophy staff/graduate seminar at the University of York, written without knowledge of any of the above publications and motivated by a concern to fill in what seemed to be an expressive gap in the logical vocabulary of (e.g.) English. A later version (‘Towards a Non-Truth-Functional Account of Logical Subtraction’) was presented at the annual conference of the Australasian Association of Philosophy in Wollongong, 1979 and a still later version (‘Logical Subtraction: Problems and Prospects’) to a University of London philosophy colloquium (University College, London) in 1981. These papers were not published, and the current incarnation of this material is in a section on logical subtraction in the ‘And’ chapter of a book in preparation (working title: *The Connectives*), on some of which the present discussion is based.

new connective ‘ $-$ ’ is ruled out by a glance at lines 3 and 4 of the truth-table, since distinct values would have to be returned for the same pair of arguments (namely  $\langle F, F \rangle$ ):

$(p \wedge q) - p$	$q$
T T T    T	T
T F F    T	F
F F T    F	T
F F F    F	F

Returning to the general case, with A and B for formulas of arbitrary complexity but to be subjected to some further conditions, we might, in the absence of a truth-functional account, at least insist that provably equivalent formulas of the logic to be extended should be intersubstitutable in the scope of the new connective (in which case we describe the new connective as *congruential* <sup>21</sup>), which is to be governed, possibly *inter alia*, by some such rule as the

**Subtraction Rule**                      *From:*  $A \wedge B \dashv\vdash C$                       *to:*  $B \dashv\vdash C - A$

Here we may for definiteness take the ‘ $\dashv\vdash$ ’ to indicate some extension of the consequence relation associated with classical propositional logic; the problems with unconstrained appeal to such a principle are not solved by passing to any seriously proposed rival to classical logic, so it is simpler to ignore non-classical logics in discussing them. (And of course ‘ $D \dashv\vdash E$ ’ abbreviates ‘ $D \vdash E$  and  $E \vdash D$ ’.<sup>22</sup>) The basic problem is of course one of conservative extension. As Jaeger pointed out (in ‘Action and Subtraction’), the passage from a consequence relation  $\vdash_0$  to the least congruential  $\vdash \supseteq \vdash_0$  will be non-conservative unless the original  $\vdash_0$  is amongst those consequence relations  $\vdash$  satisfying (i.e. ‘admitting’) the following:

**Cancellation Rule.** *From:*  $A \wedge B_1 \dashv\vdash A \wedge B_2$                       *to:*  $B_1 \dashv\vdash B_2$

But, as we see, taking (for example) A as  $p$ ,  $B_1$  as  $p \wedge q$  and  $B_2$  as  $q$ , this Cancellation Rule is not admissible for classical—or any other plausible—propositional logic taken as  $\vdash_0$ . Thus the Subtraction Rule above would yield a non-conservative (in fact, an inconsistent) extension in this case, since it would allow us to ‘subtract’ A from  $A \wedge B_1$  as well as from  $A \wedge B_2$ , to get supposedly equivalent results (by congruentiality). In the present instance, this would assert an equivalence between  $p$  and  $p \wedge q$ . Thus if anything like the

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<sup>21</sup> We also describe a consequence relation as congruential when (according to that consequence relation) every connective (in the language concerned) is congruential.

<sup>22</sup> We might add that since these are statements that a consequence relation holds rather than sequents whose provability represents that relation as holding, the above is not strictly speaking a rule, but (at best) a metalinguistic statement about the admissibility of a certain sequent-to-sequent rule. Our present concerns to not justify the introduction of the notation that would be needed for ‘de-confusing’ the description. The ‘at best’ is included because when, we below, we impose conditions about the consistent of various formulas, we are invoking hypotheses about the unprovability rather than the provability of certain sequents, and so are even further from any technically correct use of the term ‘rule’; the present casual usage is far from unprecedented, however – cf. talk of the ‘rule of disjunction’ in modal logic. Finally, we point out that, given congruentiality, closure under the Subtraction Rule is equivalent to its holding for all A and B that  $(A \wedge B) - A \dashv\vdash B$ ; we emphasize the rule formulation for parity with the Cancellation Rule.

Subtraction Rule is to be employed, it must be subjected to some constraining condition.

It has occurred naturally to several people that the required condition is some kind of independence condition. A very weak version of such a condition is suggested by the following passage from Frank Jackson:<sup>23</sup>

It sometimes seems to be thought that we can sidestep the question of whether 'sees' has 'success grammar' or 'existential import', by arguing as follows: Let us grant that 'see' as used in current English licenses inferring 'D exists' from 'S sees D'. But, for various reasons, this usage is philosophically inconvenient; hence we should conduct our discussion in terms of 'see\*', where 'see\*' means just what 'see' means, except that 'S sees\* D' does not entail 'D exists'. There is, however, a fundamental problem with such a procedure. Consider someone writing on the secondary qualities who observes that 'X is red' entails that X is coloured, and decides to introduce the term 'red\*' to mean precisely what 'red' means except that 'X is red\*' does not entail that X is coloured. The question such a procedure obviously raises is whether the deletion of the entailment to 'X is coloured' leaves anything significant behind. And it is hard to see how to settle this question other than by considering whether 'X is red' may be analyzed as a conjunction with 'X is coloured' as one conjunct and some sentence, P, not entailing 'X is coloured' as the other. If it can, 'X is red\*' means P; if not, 'red\*' has no consistent meaning at all.

The lack of any immediate intelligibility to subtracting 'X is coloured' from 'X is red' was also noted in Jaeger, 'Action and Subtraction', which distinguishes several notions of independence that might be relevant. Our quotation from Jackson makes reference to "considering whether 'X is red' may be analyzed as a conjunction with 'X is coloured' as one conjunct and some sentence, P, not entailing 'X is coloured' as the other." Setting to one side any difficulties about the problematic notion of analyzability appealed to here, by working with the simpler notion of logical equivalence, the suggestion would seem to be that for 'X is coloured' to be subtractible from 'X is red', we must find some equivalence between a conjunction with conjuncts is 'X is red' and P, for some choice of P which *does not entail* 'X is coloured'.<sup>24</sup> This kind of 'unilateral non-entailment' is a notion of independence familiar from discussions of the independence of axioms. A stronger requirement would be to 'bilateralize' the non-entailment condition, so that neither conjunct is to follow from the other. Still stronger (what Jaeger calls 'independence<sub>1</sub>') would be to add that the two conjuncts should also be compatible – that neither should entail the negation of the other. Strongest of all, while we are moving in this direction, would be a condition of complete logical independence: that as well as being independent in the previous sense, the two conjuncts should not be 'subcontraries' (the negation of one should not entail the other: Jaeger calls this 'independence<sub>2</sub>').

To see that none of these notions of independence yields a satisfactory version of the Subtraction Rule above (when the A and B appearing there are required to be

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<sup>23</sup> F. C. Jackson, *Perception*, Cambridge University Press, Cambridge 1977; the following quotation is from pp.4–5.

<sup>24</sup> As we shall see presently, this condition is far too weak to play the role expected of it; while I shall be suggesting a replacement, Jackson's own response (personal communication) is to stress the extra force of the demand of analyzability—here set aside—over the mere demand of logical equivalence.

independent in whatever sense is at issue), it suffices to demonstrate this for the strongest such notion: complete logical independence. As above, the Subtraction Rule, constrained so that A and B are completely independent, will yield a conservative extension only if the consequence relation extended satisfies the Cancellation Rule similarly constrained, which is to say: subject to the twofold condition that the formulas A and  $B_1$  exhibited in the above schematic formulation are completely logically independent, and that also the formulas A and  $B_2$  are completely independent. But it is not hard to find counterexamples revealing that even this constrained version of the Cancellation Rule is not admissible in classical propositional logic. Take, for instance  $A = p$ ,  $B_1 = q$ ,  $B_2 = p \leftrightarrow q$ . (We need to use  $p \leftrightarrow q$  rather than  $p \rightarrow q$ , in order to secure failure of subcontrariety with  $p$ .)

Now, there is a stronger condition that this strongest of all notions of logical independence available to us, which is obtained by combining two conditions on the Cancellation Rule, (i) that the formula A be consistent, and (ii) that the formulas  $B_1$  and A have no propositional variables in common, and nor do  $B_2$  and A.<sup>25</sup> As is clear from this formulation, we are thinking only of the setting of propositional logic; since that is where the counterexample just given arises, this should be instructive enough. To see that the Cancellation Rule is satisfied in the case of classical propositional logic, when subjected to the two-part condition just adumbrated, suppose that the condition is met for a particular A,  $B_1$ , and  $B_2$ , for which the conclusion is incorrect, say (without loss of generality) because  $B_2$  is not a consequence of  $B_1$ . We must show that the premiss is incorrect: the conjunctions  $A \wedge B_1$  and  $A \wedge B_2$  cannot be equivalent. Since  $B_2$  doesn't follow from  $B_1$  there is a boolean valuation<sup>26</sup> verifying  $B_1$  but falsifying  $B_2$  (call it the first valuation), and since A is *ex hypothesi* consistent there is a boolean valuation verifying A (the second valuation). Now, as A shares no variables with  $B_1$  or  $B_2$ , there is a boolean valuation which is like the first valuation on formulas constructed out of the variables occurring in  $B_1$  or  $B_2$  or both, and like the second valuation on those constructed out of variables occurring in A. Note that for this new valuation, V, say, since  $V(B_1) = T$ ,  $V(B_2) = F$ , and  $V(A) = T$ , we have  $V(A \wedge B_1) = T$  while  $V(A \wedge B_2) = F$ , which implies that  $A \wedge B_1 \not\equiv A \wedge B_2$ , since V is a boolean valuation. The Subtraction Rule needs therefore to be constrained by the corresponding (i)–(ii) condition rather than merely by the complete independence condition considered earlier, in order to keep the extension conservative. That is, the condition on A and B is that: (i) A is consistent and (ii) A and B have no propositional variable in common.

Commenting, in the last of the sources mentioned in note 20, on the position at which we have now arrived, I was content to say the following by way of a gloss on the variable-disjointness condition (part (i) of the twofold condition), as it bears on the previous counterexample in which A was  $p$  and  $B_2$  was  $p \leftrightarrow q$ : although these formulas are completely “independent, so that (as far as boolean valuations are concerned) the

<sup>25</sup> The two conditions here suffice, as we shall see, for the Cancellation Rule to be satisfied, though we should need to supplement them by a third to have here a notion of independence which implies complete logical independence in the sense isolated before, namely: (iii) that the negation of A should be consistent. In Section 4 this supplemented condition will emerge under the title of ‘ultra-independence’.

<sup>26</sup> A truth-value assignment respecting the usual truth-table requirements on how compounds’ truth-values are determined by their components’, that is.

truth-values they receive are independent, the ways in which they get these truth-values are not independent. This is reflected syntactically by the fact that (or: is a reflection of the syntactic fact that) they have a propositional variable in common.” It seems to me now, however, that one cannot simply leave it at that: to do so would be to give the impression that matters are all too syntactical. If we think of the language as an interpreted propositional language, distinct sentence letters being distinct atomic sentences, then a mere change in the choice of what to take as the class of atomic sentences can lead, notoriously, to a language intertranslatable with the original but on which decisions sensitive to such matters as variable-disjointness (‘atom-disjointness’) come out differently.<sup>27</sup> That strongly suggests that such decisions were so highly language-dependent that nothing of any substance could possibly hang on them. In particular suppose that we have a propositional language,  $L_1$ , with atomic sentences  $p$  and  $q$ , and another,  $L_2$ , with atomic sentences  $p$  and  $r$ ,  $p$  meaning the same thing in both languages, and  $r$  in the second expressing what would be expressed by  $p \leftrightarrow q$  in the first. This shows how to translate  $L_2$  into  $L_1$ , but we can translate back by sending the  $L_1$  formula  $q$  to the  $L_2$  formula  $p \leftrightarrow r$ .<sup>28</sup> Then what is in  $L_1$  a consistent conjunction with variable-disjoint conjuncts,  $q \wedge p$ , a formula which accordingly allows (by the twofold condition above) the subtraction of  $p$  to yield something equivalent to  $q$ , becomes in  $L_2$  the formula  $(p \leftrightarrow r) \wedge p$ , which does not pass the variable-disjointness part of the two-part test. However, this formula is equivalent to a conjunction that does pass the test, namely,  $r \wedge p$ . But of course, when we apply the Subtraction Rule to this equivalence, we end up concluding that  $(r \wedge p) - p$ , and therefore, by congruentiality,  $((p \leftrightarrow r) \wedge p) - p$ , is equivalent to  $r$ . Here we have an exceedingly unwelcome language-dependence: performing a subtraction in  $L_1$  gives one result, while performing the corresponding subtraction in the expressively equivalent language,  $L_2$ , gives a different result. (It is true that the admissibility of the Cancellation Rule shows a kind of uniqueness for the results of applying the Subtraction Rule: but this is only uniqueness within a single language, with its allotment of sentence letters.) What would be desirable would be some indication that at most one of the two languages just contrasted provides a suitable environment in which to invoke our twofold condition on subtraction. We address this topic in the following section, after closing with some remarks which assume the problem of excessive language-dependence to be soluble.

There is a conspicuous difference in the intelligibility, as witnessed by the reactions of such ‘informants’ as I have consulted, of, on the one hand, something like  $(p \wedge q) - p$ ,

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<sup>27</sup> The notoriety (of potential unwanted sensitivity to choice of primitives) comes from such discussions as Nelson Goodman’s ‘grue’ example, though more directly bearing on the way I have put the point here, the most important reference is perhaps David Miller, ‘Popper’s Qualitative Theory of Verisimilitude’, *British Journal for the Philosophy of Science* **25** (1974), 166–177.

<sup>28</sup> This is essentially the example used in the Arizonan/Minnesotan discussion in Section 3 of Miller *op. cit.*; the intertranslatability point relies on the equivalence in classical logic of  $(A \leftrightarrow B) \leftrightarrow B$  with  $A$ . (By contrast with the earlier point about the non-conservativeness of an extension with the unrestricted Subtraction Rule, the present point is highly sensitive to the choice of classical logic.) As Miller remarked in a later paper (‘Verisimilitude Redeflated’, *British Journal for the Philosophy of Science* **27** (1976), 363–381), the first person to make such a point was Max Black, in whose *A Companion to Wittgenstein’s Tractatus* (Cambridge University Press, Cambridge 1964) it appears on p.46, though the example there is slightly more complicated than Miller’s. We describe Miller’s example in Section 5 below.

and, on the other hand something like  $p - (p \vee q)$ . It wouldn't do to react to this by saying that while conjuncts are (intuitively) subtractible from conjunctions containing them, disjunctions are not similarly subtractible from their disjuncts, notwithstanding the fact that the material to be subtracted follows logically from what it is to be subtracted from in both cases alike. That would violate the desideratum of providing a congruential account, since we could rephrase the problematic ' $p - (p \vee q)$ ' as ' $((p \vee q) \wedge (p \vee \neg q)) - (p \vee q)$ ', so that it had the desired 'conjunction *minus* conjunct' form: but we want to say that these two subtraction expressions, the original and the rephrasal, are to be treated in the same way. If either is intelligible, so is the other. Of course, there is the question of what to say about the 'intuitively unsubtractible' cases, which include not only that raised here, but also cases in which what is to be subtracted does not even follow from what it is to be subtracted from. I would be happy to have a theory which works for the clear cases decide these by whatever principles fit in best with its successful treatment of the clear cases. We are not here in the business of providing a theory – for example, a compositional semantics for logical subtraction – just of focussing on one issue: the relevant notion of independence for a treatment which provides a Subtraction Rule as above. Returning to our example, we find that although we can reformulate ' $p - (p \vee q)$ ' as ' $((p \vee q) \wedge (p \vee \neg q)) - (p \vee q)$ ', so that the 'subtrahend' is one of the two conjuncts of a conjunction, this conjunction conspicuously fails to meet the variable-disjointness condition, and there is no embarrassing obligation to accept that  $p - (p \vee q)$  is equivalent to  $p \vee \neg q$ . In fact it is not hard to see that there is no consistent formula variable-disjoint from  $p \vee q$ , which, when conjoined with  $p \vee q$  has  $p$  as a consequence.<sup>29</sup>

The distinction between subtracting a conjunct from a (suitably independent) conjunction and trying to subtract a disjunction from a disjunct is, incidentally, reflected by the quantificational analogues of these connectives. Once it is explained to them that logical subtraction is to be a device for removing requirements for the truth of that from which the subtrahend is being subtracted, people have no difficulty making immediate sense of *Everyone is  $\phi$  - Arthur is  $\phi$* , at least if ' $\phi$ ' is thought of as expressing some intrinsic property; the upshot is that everyone, with the possible exception of Arthur, has that property.<sup>30</sup> An interesting side benefit of this example is that if we imagined logical subtraction added to a first-order language *without* identity, then we should have here a simple example of increased expressive power, since there is no way in the absence of identity (and of course subtraction), to say what  $\forall x(x \neq a \rightarrow \phi x)$  says. However, our main point in giving the example is to contrast it with the dual case:

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<sup>29</sup> For let B be such a formula. Then B is variable-disjoint from  $p \vee q$  and so also from  $p$ , even though  $B, p \vee q \vdash p$ ; the inconsistency of B then follows by what D. J. Shoesmith and T. J. Smiley (*Multiple-Conclusion Logic*, Cambridge University Press, Cambridge 1978) call the Cancellation Property (mid-page 272): 'Cancellation' here refers to simply deleting a formula from a consequence statement, rather than the 'canceling-out' we have in mind in the Cancellation Rule above. While we are addressing such terminological matters, it is worth pointing out that the 'Canceling Out' fallacy described in P. T. Geach, *Reference and Generality* (Cornell University Press, Ithaca 1968) does indeed have as a special case the invoking of an *unrestricted* version of our Cancellation Rule – whence the need for the restrictions, in fact.

<sup>30</sup> We have to be careful about people who volunteer the response "Everyone except Arthur is  $\phi$ ", since they may understand this as implying that Arthur is *not*  $\phi$ . In that case, we have to re-explain that subtracting (propositional) content is removing your commitment to that content, rather than incurring a commitment to its negation.

*Arthur is  $\phi$  – Someone is  $\phi$* , which, on my informal surveys, manages (like the Jaeger-Jackson example ‘X is red – X is coloured’) to draw nothing but expressions of puzzlement. We return to the matter of ‘intuitively unsubtractible’ cases at the end of the following section.

#### 4. Mereological Assistance On the Independence Issue

The question left over from our discussion of the restriction in terms of variable-disjointness on the Subtraction and Cancellation Rules in the previous section is whether this can be given a more robust justification than that mentioned there – involving in particular a defence against the charge that a mere change of language will change inappropriately what is deemed to be the result of subtracting something from something else. Translating the something and the something else between languages disrupts the functioning of the Subtraction Rule because of its sensitivity to the question of which sentences are atomic. It is to the boolean algebra of parts that we look for assistance.

The role of mereology in discussions of logical subtraction is not new. Hudson (*op. cit.*) alleged that the earlier discussion by Jaeger confused logical with mereological relations. Specifically, Hudson is defending the proposal that logical subtraction is none other than the truth-functional connective of converse material implication (on the grounds that  $A - B$  should be the logically weakest statement which together with  $B$  yields  $A$  as a consequence, and thus expresses the same proposition as  $B \rightarrow A$ ), against Jaeger’s demand that  $A - B$  should be completely logically independent of  $B$ , whereas  $B \rightarrow A$  (or “ $B \supset A$ ”, in the notation used by Hudson) and  $B$  are subcontraries:

Now, do we have intuitive grounds for requiring the stricter independence of  $Q$  and  $P - Q$ ? Jaeger apparently thinks so, and I hypothesize that that his opinion is based in part on an analogy with physical subtraction; for each stone in a pile is quite independent of each other stone, and disjoint subpiles are independent of each other. Thus a logical atomist, thinking of the atomic facts  $Q$  and  $R$  as being like distinct stones and of the molecular fact  $Q \& R$  as being like a two stone pile, might naturally want  $(Q \& R) - Q$  to be  $R$  rather than  $Q \supset (Q \& R) [= Q \supset R]$ . (...) But to reason thus is to forget the very analogy between propositions and subpiles of stones on which we have been relying. According to this analogy, what corresponds to strong logical independence is the failure of either of two subpiles to be wholly contained in *or wholly excluded from* the other. Thus two piles would have to overlap somewhat (...) in order to correspond to two strongly independent propositions. On the other hand, two subpiles which did not overlap at all, and which might thus be said to be “physically independent”, would correspond to two propositions which, while weakly independent, were subcontraries.<sup>31</sup>

On the first analogy between statements and piles of stones that may come to the reader’s mind, this passage will seem confusing. This is the analogy according to which the stones are possible worlds and the pile of stones to which a statement corresponds is simply the proposition (in the sense of Section 2) it expresses. Confusion would then result because the overlapping of piles then corresponds to the joint consistency of the

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<sup>31</sup> Hudson, *op. cit.*, p.132. Weak and strong independence are Jaeger’s independence<sub>1</sub> and independence<sub>2</sub>, as explained in Section 3; thus the latter is complete logical independence.

two statements, to their not being contraries, in other words, whereas Hudson links such overlap to their not being subcontraries. So to make the above passage reasonable, the “pile of worlds” corresponding to a statement should not comprise those worlds at which it is true – the proposition expressed by the statement – but those at which it is false (what is sometimes called the *content* of the statement<sup>32</sup>). Either way, it is clear that disjointness between subsets of  $W$  is no place to look for any notion of independence, since such disjointness amounts to the kind of exclusion of one statement by another—whether of  $B$  by  $A$  (incompatibility or ‘contrariety’), or of  $B$  by the negation of  $A$  (subcontrariety), or of the negation of  $B$  by  $A$  (entailment or logical implication): exactly the relationships independence is supposed to rule out.

In Section 3, however, we did find a link between disjointness and independence, since orthogonality of subject matters is precisely a notion of complete independence ( $M$  and  $N$  being orthogonal when every proposition constituting a block of  $M$  is thus independent of every block of  $N$ ), and the principle (Recomb.) we adapted from Lewis said that disjoint regions determine orthogonal subject matters. Thus it is disjointness in the algebra of parts rather than in the algebra of propositions that should be considered as providing a candidate for logical independence. In Section 3, we found an unfortunate language-sensitivity, when we considered the propositional languages there called  $L_1$  and  $L_2$ , of the result of ‘subtracting’  $p$  from a conjunction we could write in the first of these languages as ‘ $q \wedge p$ ’ (alternatively, in the non-variable-disjoint form suggested by the translation from the other language, as ‘ $(p \leftrightarrow q) \wedge p$ ’) and from a conjunction we could write in the second language as ‘ $r \wedge p$ ’, or again (as suggested by the given translation, ‘ $(p \leftrightarrow r) \wedge p$ ’): the results were respectively  $q$  and  $r$  — formulas which are *not* equivalent under the translations provided. What I am going to suggest is that there is a certain interesting property of interpreted propositional languages which not both of our two languages here can possess, and that logical subtraction can be understood in part in terms of this property. I’ll call the property ‘segregation’. To specify an interpretation for such a language I am going to suppose that we specify a set  $W$  of worlds and a set  $\mathbf{M}$  of parts understood as functions with domain  $W$  as in Section 2, and interrelated by the various numbered principles of that section, and a function  $\| \cdot \|$  assigning to each formula  $A$  a subset of  $\|A\|$  of  $W$ , in accordance with the usual recursive conditions for the truth-functional connectives. Such a language, paired with its interpretation, we will describe as *segregated* if its atomic formulas, which – to

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<sup>32</sup> This usage, reminiscent of Popper, of the term ‘content’ is found for instance in Lewis, ‘Statements Partly About Observation’; in the unpublished work mentioned in note 20, I used this idea of content to examine the suggestion that we explain  $A - B$  for arbitrary  $A, B$ , by stipulating that the content of  $A - B$  is to be the content of  $A$  ‘minus’ (in the sense of relative complementation or set-theoretic difference) the content of  $B$ : as the reader may check, this makes the proposition expressed by  $A - B$  the same as that expressed by  $B \supset A$ , confirming that it is this is the correspondence that Hudson has in mind. (Of course, in view of what was said at the start of Section 3 above, I think any such truth-functional account is a non-starter amongst candidate explications of logical subtraction. A more thorough discussion of this matter would need to consider the role of implications other than that of classical propositional logic, as having converses worth considering as candidates for logical subtraction. In this connection, it is worth recalling that the theory of *BCK*-algebras originated with the observation that arithmetical subtraction and set-theoretic difference share many formal similarities with the converse of implication in some weaker logics. See the survey article by K. Iséki and S. Tanaka, ‘An Introduction to the Theory of *BCK*-Algebras’, *Mathematica Japonica* **23** (1978), 1–26.)

emphasize the pairing – we now call sentence-letters rather than propositional variables, collectively satisfy the condition (4.1) below. To formulate this condition we need the notion of the *habitat* of a formula A (under a given interpretation), which is to be the  $\sqsubseteq$ -smallest element  $m$  of  $\mathbf{M}$  such that A is entirely about  $\sigma(m)$ . Every formula has a habitat in this sense, because every formula is (at worst) entirely about  $\blacksquare$  (i.e., entirely about  $\sigma(\blacksquare)$ : we generally drop the ‘ $\sigma$ ’ from now on), and whenever A is entirely about two regions, it is entirely about their mereological intersection, by (2.9). Writing  $hab(A)$  for the habitat of A, we can give our segregation condition:

(4.1) *If A and B are distinct atomic formulas, then  $hab(A) \sqcap hab(B) = \blacksquare$ .*

If  $p_1, \dots, p_k$  are the only sentence letters in A, then it follows from (2.6) and (2.14) that A is entirely about  $hab(p_1) \sqcup \dots \sqcup hab(p_k)$ , and therefore that  $hab(A) \sqsubseteq (hab(p_1) \sqcup \dots \sqcup hab(p_k))$ . Analogous reasoning for another formula B, together with the distributive law (for  $\mathbf{M}$ ), then allows us to derive (4.2) from (4.1):

(4.2) *If A and B are variable-disjoint formulas, then  $hab(A) \sqcap hab(B) = \blacksquare$ .*

The condition (4.2) is a generalized form of the basic segregation condition (4.1), which perhaps emphasizes the degree of idealization involved in the assumption that we are working with a segregated language. No two atomic sentences of the language can be used to record something intrinsic to the state of the same non-empty spatial region, or of two overlapping regions. (We can’t say that no distinct atomic formulas are entirely about any common region, since if they are entirely about two regions then any region of which the fusion of those two is a part will be a region both are entirely about: hence the focus, in the notion of habitat, on the smallest region a formula is entirely about.) Idealized or not, the present setting does throw some light on the language-dependence worry for the Subtraction and Cancellation Rules of Section 3. The discussion surrounding Figure 1, in Section 2, provides us with a  $W$  and  $\mathbf{M}$  with which to interpret the language  $L_1$  from Section 3, in such a way that the result is a segregated interpreted language. Recall that  $W = \{1,2,3,4\}$  and  $\mathbf{M} = \{\blacksquare, m, n, \square\}$ ; we take over the relation  $\gg$  from that discussion with its associated  $m$ - and  $n$ -states there called *on* and *off*. If we set  $\|p\| = \{1,2\}$  and  $\|q\| = \{1,3\}$ , then we have  $hab(p) = m$  and  $hab(q) = n$ ,  $w \in \|p\| \Leftrightarrow m(w)$  is *on*, and  $w \in \|q\| \Leftrightarrow n(w)$  is *on*, for all  $w \in W$ . As is required by (4.1), the two atomic sentences of  $L_1$  are assigned mereologically disjoint habitats, so the subject matters determined by these two regions are orthogonal, in accordance with (Recomb.). But these subject matters are respectively  $\wp?$  and  $\zeta q?$ , and to say that any two such 2-block partitions  $\wp A?$ ,  $\zeta B?$ , are orthogonal is just to say that the statements (interpreted sentences) A and B are completely logically independent. This is how the complete independence of the different atomic sentences of a segregated language is secured. But what is secured is weaker than what in this case secures it – the disjointness of the habitats of these sentences – and it is precisely here that things go astray if we pass from  $L_1$  to  $L_2$  by the translation described in Section 3. That translation required that we interpret the latter language’s two atomic sentences  $p$  and  $r$  in such a way – calling the new interpretation  $\| \|^*$  – that  $\|p\|^* = \|p\|$  and  $\|r\|^* = \|p \leftrightarrow q\|$ .  $L_2$ ’s  $p$  and  $r$  were again completely independent, which in the current setting reflects the orthogonality of  $\wp?$

and  $\wp \leftrightarrow q?$  (what we could call  $\wp p?^*$  and  $\wp r?^*$ ); but although these are orthogonal subject matters, they—by contrast with  $\wp p?$  and  $\wp q?$ —are not orthogonal *part-based* subject matters. As we remarked at the end of Section 2, 14|23, which is the current  $\wp p \leftrightarrow q?$ , is not the subject matter determined by any region in  $\mathbf{M}$ . To be sure,  $p \leftrightarrow q$  is entirely about  $m \sqcup n$  (*alias*  $\blacksquare$ ), but this is not a region disjoint from  $m$ , the smallest region that  $p$  is entirely about (*alias* the habitat of  $p$ ). So  $L_2$ , as interpreted here, is not a segregated language. In a segregated language, such as  $L_1$ , the independence of distinct atomic sentences reflects the disjointness of the parts of reality they address. Independence of aspects of reality – orthogonality of subject matters in general – is a weaker condition, as is evident in this example, than this mereologically grounded independence of specifically part-based subject matters. Of course the situation is symmetrical *vis-à-vis*  $L_2$  and  $L_1$ : if we start off assuming that it is  $L_2$  that is segregated (and exhibiting  $\mathbf{M}$ ,  $W$ , and  $\| \|$  with respect to which this is so) then we shall have to conclude that  $L_1$  is not. In either case, there is no possibility of a division of reality into parts in such a way that a sentence A (in both of our cases,  $p$ ), a sentence B ( $q$  in the first case,  $r$  in the second) and the sentence  $A \leftrightarrow B$ , all get to be about pairwise disjoint parts.<sup>33</sup> It would be a mistake to say, by way of objection, that just as—by changing the atomic formulas—we can begin with another way of dividing up logical space, so physical space can be carved up in different ways: for the point we have been making holds with respect to any such carving up. That is, relative to any given division into parts, we cannot have pairwise disjoint habitats for  $p$ ,  $q$  (*alias*  $p \leftrightarrow r$ ), and  $p \leftrightarrow q$  (*alias*  $r$ ).

If we relax the segregation condition (4.1) we can still make a suitably modified version of this last point. We could allow distinct atomic sentences to have the same region as habitat (allowing, thus, for the acknowledgement of several logically independent ways for a given region to be) and code this ‘co-regionality’ syntactically. Sticking, for simplicity, with the two-basic-regions case, we could have  $m$ -habitat atoms  $p_1, p_2, p_3$  and  $n$ -habitat atoms  $q_1, q_2$  (say). Then if we say that all the  $p$ -atoms are of one *type* and all the  $q$ -atoms of another, we can revise (4.1) to read “if A and B are atomic sentences of distinct types, then they have disjoint habitats”, and discuss such – we might call them – *weakly* segregated (interpreted) languages. There would be an analogue of (4.2) in the form: if no atomic sentence in A is of the same type as any atomic sentence in B, then  $hab(A) \sqcap hab(B) = \square$ . The restriction on the Subtraction Rule, for application in connection with a weakly segregated language:

$$\text{From: } A \wedge B \dashv\vdash C \qquad \text{to: } A \dashv\vdash C - B$$

will then have to be not simply that A and B are variable-disjoint, but that no atomic sentence in A is of the same type as any atomic sentence in B. We continue the present discussion under the stronger assumption of segregation *à la* (4.1), however.

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<sup>33</sup> This point is structurally analogous to Max Black’s point (*op. cit.*, p.47) about the constitution of Wittgenstein’s atomic facts out of objects. Note that there has been no resort to facts – whether of the eccentrically Tractarian variety or of a more familiar sort – in the present account; my impression is that unless something like the mereological emphasis of our discussion is injected into the discussion of facts, issues of independence and orthogonality will remain unresolved and we might as well have remained at the linguistic level.

How are the habitats of compounds related to the habitats of their components? If # is a  $k$ -ary truth-functional mode of composition then we can at least say this much (for reasons given *à propos* of the special case in which the  $A_i$  were distinct atomic formulas above: see the discussion between (4.1) and (4.2)).

$$(4.3) \quad \text{hab}(\#(A_1, \dots, A_k)) \sqsubseteq (\text{hab}(A_1) \sqcup \dots \sqcup \text{hab}(A_k))$$

The ‘ $\sqsubseteq$ ’ cannot be converted to a ‘=’, however, as the following example shows. Let  $\text{hab}(p_i) = m_i$  (so by (4.1),  $m_i$  and  $m_j$  are disjoint when  $i \neq j$ ). Then  $\text{hab}(p_1) = m_1$ ,  $\text{hab}(p_1 \vee p_2) = m_1 \sqcup m_2$ , but  $\text{hab}(p_1 \wedge (p_1 \vee p_2)) = \text{hab}(p_1) \neq m_1 \sqcup (m_1 \sqcup m_2) = m_1 \sqcup m_2$ . A variation on this example shows that there is no general principle associating, recursively, with an arbitrary # a function  $f: \mathbf{M} \rightarrow \mathbf{M}$  for which (4.4) holds:

$$(4.4) \quad \text{hab}(\#(A_1, \dots, A_k)) = f(\text{hab}(A_1), \dots, \text{hab}(A_k))$$

For, taking # as  $\wedge$ , we have  $\text{hab}(p_1 \wedge (p_1 \vee p_2)) = m_1$  whereas  $\text{hab}(\neg p_1 \wedge (p_1 \vee p_2)) = \text{hab}(\neg p_1 \wedge p_2) = m_1 \sqcup m_2$ , even though  $\text{hab}(p_1) = \text{hab}(\neg p_1)$ .

The above discussion makes several assertions without justifying them. For instance, we have just claimed that  $\text{hab}(p_1) = \text{hab}(\neg p_1)$ . In this case, we have as a justification of the more general claim that  $\text{hab}(A) = \text{hab}(\neg A)$ , for any formula A: this follows from the fact that a formula and its negation are entirely about precisely the same regions – a consequence of the definition of ‘entirely about’. The claims (i) that  $\text{hab}(p_1 \vee p_2) = m_1 \sqcup m_2$ , which is to say that  $\text{hab}(p_1 \vee p_2) = \text{hab}(p_1) \sqcup \text{hab}(p_2)$ , and (ii) that  $\text{hab}(\neg p_1 \wedge p_2) = m_1 \sqcup m_2$ , which is to say that  $\text{hab}(\neg p_1 \wedge p_2) = \text{hab}(\neg p_1) \sqcup \text{hab}(p_2)$ , are less straightforward. We address the case of (i) first. We show, more generally, that if A and B are any two formulas which are, as we shall put it, ‘ultra-independent’ formulas, then  $\text{hab}(A \vee B) = \text{hab}(A) \sqcup \text{hab}(B)$ ; our (i) is the special case in which A and B are distinct sentence letters. In saying that A and B, two formulas of an interpreted language (so that we have a fixed  $\mathbf{M}$ ,  $W$ , and  $\| \cdot \|$  in mind) are *ultra-independent* just in case A and B are variable-disjoint, and neither  $\zeta A?$  nor  $\zeta B? = \mathbf{1}$ .<sup>34</sup> Since we already know that  $\text{hab}(A \vee B) \sqsubseteq \text{hab}(A) \sqcup \text{hab}(B)$  even without the conditions on A and B (by (4.3)), what has to be shown is that under those conditions, we also have  $\text{hab}(A) \sqcup \text{hab}(B) \sqsubseteq \text{hab}(A \vee B)$ . Since the hypotheses on A and B treat those formulas alike, it will suffice to show that  $\text{hab}(A) \sqsubseteq \text{hab}(A \vee B)$ . We prove this as (4.6) below; first we need a Lemma:

(4.5) If for a formula C and a region  $m \in \mathbf{M}$  we have  $m \sqsubseteq \text{hab}(C)$  and  $m \neq \square$ , then there exist  $u, v \in W$  with  $u \gg_{m'} v$ , and C having different truth-values in  $u$  and  $v$ .

We could think of (4.5) as saying that the truth-value of a formula is potentially sensitive to how things are with any non-empty part of its habitat, since it tells us that for any such part there are worlds differing only in respect of that part (being duplicates over the complementary region) and returning different truth-values to the formula in question. To prove (4.5), suppose that  $m \sqsubseteq \text{hab}(C)$  and there do not exist  $u, v \in W$  as

<sup>34</sup> This last part of the definition is equivalent to saying that  $\|A\| \neq W$  and  $\|A\| \neq \emptyset$ , and likewise for B.

promised; in that case  $C$  is entirely about  $m'$  and so  $hab(C) \sqsubseteq m'$ ; thus  $m \sqsubseteq m'$ , implying that  $m = \square$  and thereby establishing (4.5). We are now in a position to attend to:

(4.6) If  $A$  and  $B$  are ultra-independent then  $hab(A) \sqsubseteq hab(A \vee B)$ .

We shall assume, for a contradiction, that for  $A$  and  $B$  as described,  $hab(A) \sqcap (hab(A \vee B))' \neq \square$ ; we denote this—the non-empty part of  $hab(A)$  not included in  $hab(A \vee B)$ —by  $l$ ; we also denote  $hab(A) \sqcap (hab(A \vee B))$  by  $m$ , and  $hab(B)$  by  $n$ . Thus  $hab(A) = l \sqcup m$ , and (by (4.3))  $hab(A \vee B) \sqsubseteq m \sqcup n$ , and (by (4.2))  $(l \sqcup m) \sqcap n = \square$ . To denote states of the pairwise regions  $l, m, n$ , we use the corresponding lower case Greek letters,  $\lambda, \mu, \nu$ , possibly ornamented by asterisks. (Think, for example, of  $\mu$  and  $\mu^*$  as labels for two from amongst the various blocks of the subject matter  $\sigma(m)$ , *alias* two  $\gg_m$ -equivalence classes.) We use the notation ' $\gg$ ' (heuristic reading: “secures”) in such a way that, for instance, “ $\lambda, \mu \gg C$ ” means that for all  $w \in W$ , if  $l(w)$  is in state  $\lambda$  and  $m(w)$  is in state  $\mu$ , then  $C$  is true at  $w$ ; “ $\mu^*, \nu \gg C$ ” means that for all  $w \in W$ , if  $m(w)$  is in state  $\mu^*$  and  $n(w)$  is in state  $\nu$ , then  $C$  is true at  $w$ ; and so on.<sup>35</sup> Now since  $l$  is a non-empty part of  $hab(A)$ , by (4.5) there exist worlds which are duplicates in respect of (*inter alia*)  $m$  and  $n$ , at which  $A$  takes different truth-values. Pick such a pair of worlds and let  $\lambda$  and  $\lambda^*$  be the state of region  $l$  in, respectively, the  $A$ -verifying world and the  $A$ -falsifying world, and let  $\mu$  be the state of  $m$  in both worlds. Since  $l \sqcup m$  is  $hab(A)$ ,  $A$  is entirely about  $l \sqcup m$ , and (a) and (b), below, follow. Since the formula  $A$  and  $B$  are ultra-independent,  $B$  is false at some world (see note 34), and letting  $\nu$  be the state of  $n$  in that world (recalling that  $n = hab(B)$ ), we have (c):

$$(a) \lambda, \mu \gg A \qquad (b) \lambda^*, \mu \gg \neg A \qquad (c) \nu \gg \neg B$$

From (b) and (c) we infer that  $\lambda^*, \mu, \nu \gg \neg A$  and  $\lambda^*, \mu, \nu \gg \neg B$ , and hence that  $\lambda^*, \mu, \nu \gg \neg(A \vee B)$ . Now, since  $hab(\neg(A \vee B)) = hab(A \vee B) \sqsubseteq m \sqcup n$ , we can drop the reference, in the shape of  $\lambda^*$ , to what is going on in  $l$ , which is disjoint from the habitat of  $\neg(A \vee B)$ , and conclude that  $\mu, \nu \gg \neg(A \vee B)$ . It follows then that  $\lambda, \mu, \nu \gg \neg(A \vee B)$ . But (a) implies that both  $\lambda, \mu, \nu \gg A$  and hence  $\lambda, \mu, \nu \gg A \vee B$ . Since no world can verify  $A \vee B$  and its negation, this would imply that for no  $w \in W$  do we have  $l(w)$  in state  $\lambda$ ,  $m(w)$  in state  $\mu$ , and  $n(w)$  in state  $\nu$ : but given the existence of worlds whose respective states are as indicated, that would contradict (Recomb.), promising a single world with the said distribution of states. This concludes the proof of (4.6), and with it the argument for

(4.7) If  $A$  and  $B$  are ultra-independent then  $hab(A \vee B) = hab(A) \sqcup hab(B)$ .

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<sup>35</sup> It may be helpful to think of statements such as “ $\lambda, \mu \gg C$ ” as statements about a heterogeneous consequence relation, relating sets of the pseudo-linguistic entities represented by the Greek letters on the left to the genuinely linguistic formulas on the right; this explains the use of the term ‘thinning’ below. For a general exploration of such relations (except that multiple and empty succedents are also considered there), see my, ‘Heterogeneous Logic’, *Erkenntnis* **29** (1988), 395–435. This perspective can also be used to make precise the notion, here left at an intuitive level, of translations from one formal language to another (as in the  $L_1/L_2$  example of §3).

As mentioned above, this yields our earlier claim (i), taking A and B as distinct atomic sentences.

Claim (ii), which is similar except that B and A are respectively an atomic sentence and the negation of an atomic sentence, and has ‘ $\wedge$ ’ in place of ‘ $\vee$ ’, follows from (4.8):

(4.8) If A and B are ultra-independent then  $hab(A \wedge B) = hab(A) \sqcup hab(B)$ .

Again, the crucial step required for (4.8) is analogous to (4.6):

(4.9) If A and B are ultra-independent then  $hab(A) \sqsubseteq hab(A \wedge B)$ .

And the argument is more or less as for (4.6). We need (a) and (b) from the discussion of that case, with regions  $l, m, n$ , and states  $\lambda, \lambda^*, \mu$  as there, except with the conjunction rather than the disjunction of A with B. In place of (c), we use the datum (d) below, where  $v^*$  is the state of  $n$  in some world at which B is (not—as before—false, but) true – and since B is ultra-independent of A, there must be such worlds in  $W$  (note 34 again):

(a)  $\lambda, \mu \gg A$                       (b)  $\lambda^*, \mu \gg \neg A$                       (d)  $v^* \gg B$

(a) and (d) imply that  $\lambda, \mu, v^* \gg A \wedge B$ ; since  $hab(A \wedge B) \sqsubseteq m \sqcup n$ , we have  $\mu, v^* \gg A \wedge B$ , and hence  $\mu, v^* \gg A$ . Weakening this to  $\lambda^*, \mu, v^* \gg A$  now gives a contradiction with (b), in view of (Recomb.), as before. This establishes (4.9), and with it (4.8). It is not hard to extract—though we shall not pause to do so—a generalized version of (4.7) and (4.8), saying what they say, *à propos* of disjunction and conjunction, for the case of any ‘essentially binary’ truth-functional connective (*i.e.*, where the truth-function concerned depends on both of its arguments). It is easy to see that the restriction to essentially binary connectives is required: if we have in the language a binary connective  $\textcircled{1}$  (“first projection”) such that for all formulas A and B,  $\|A \textcircled{1} B\| = \|A\|$ , for example, then  $hab(A \textcircled{1} B) = hab(A)$  and so if A and B are distinct atomic sentences we shan’t have  $hab(A \textcircled{1} B) = hab(A) \sqcup hab(B)$ , because it won’t be the case that  $hab(B) \sqsubseteq hab(A \textcircled{1} B)$ . (The need for the restriction to ultra-independent A, B, was of course already illustrated in the discussion following (4.3).)

Because we are not concerned in this paper with defending any particular positive account of the logic of logical subtraction, we content ourselves with a brief remark as to the direction in which the above discussion leads, for the treatment of  $hab(A - B)$ . In Section 3, it was pointed out that even when B is a consequence of A, B may not be, as we put it there, intuitively subtractible from A, in that one attaches no (pre-theoretical) sense to the idea of a proposition which requires for its truth whatever is required for the truth of A with the exception of what is required for the truth of B. We return to the characterization of this distinction between the intuitively unsubtractible and the intuitively subtractible cases in terms of the present apparatus in the following paragraph. Let us for the moment simply suppose that we are dealing with A and B for which B *is* intuitively subtractible from A, and consider what might be said about  $hab(A - B)$  for this case. The obvious suggestion is that here  $hab(A - B)$  should be  $hab(A) \sqcap (hab(B))'$ : the mereological complement of  $hab(B)$  relative to  $hab(A)$ . A says something

directly about  $hab(A)$ , and indirectly (*i.e.*, in consequence) about every region with  $hab(A)$  as a part. When B says something directly about a part of  $hab(A)$ , it is this that should be deleted from the totality of what A says about  $hab(A)$ , so that  $A - B$  addresses only the remaining part of  $hab(A)$ : in other words,  $hab(A) \sqcap (hab(B))'$ . A full elaboration of this suggestion would involve explicating the idea of what a statement says about a region, and this is something that we do not propose to take up here. Even without doing so, we can throw some light on what it is about some cases in which B is not ‘intuitively subtractible’ even from an A which logically implies it.

The idea is all but explicit in the preceding remarks. Sticking with our interpreted sentential language (and for simplicity with the segregation condition (4.1) still in force), B’s being a consequence of A amounts to its being the case that  $\|A\| \subseteq \|B\|$ ; let us say that B is an *endogenous consequence* of A (or is “endogenously implied” by A) when in addition  $hab(B) \sqsubseteq hab(A)$ . An equivalent formulation of this last condition would be to say that for all  $m \in \mathbf{M}$ , if A is entirely about  $\sigma(m)$ , then B is entirely about  $\sigma(m)$ ; so we may think of endogenous implication as preservation of truth coupled with preservation of mereological aboutness. Our conjecture, consilient with the informal remarks of the preceding paragraph, is then that the intuitively subtractible consequences of a statement are precisely its endogenous consequences. One might think that these go beyond the intuitively subtractible cases in that A is always an endogenous consequence of itself, though it is hard to make intuitive sense of  $A - A$ . But what we have here is a difficulty in respect, at most, of pointfulness rather than of intelligibility. If whatever is required for the truth of A is taken away from whatever is required for the truth of A itself, then we are left with something with no requirements to satisfy in order to be true, so it scarcely calls for any extension of the basic intuitions involved to conclude that  $\|A - A\| = W$ . We turn to the asymmetry remarked on in Section 4 between the cases of conjunction-to-conjunct inferences and disjunct-to-disjunction inferences. The first point to note is that that is not an accurate description of our earlier observations. For we noted that to allow every conjunction—even restricting attention to the case in which the conjuncts were completely logically independent—to have subtractible conjuncts, would mean that  $(p \vee q)$  would be subtractible from  $p$  since the latter can be presented as such an independent conjunction in the form  $(p \vee q) \wedge (p \vee \neg q)$ . So while we want A (and B too) to be subtractible from  $A \wedge B$  when A and B are distinct atomic sentences, and under whatever is the appropriate generalization of this, that generalization should not go so far as to allow arbitrary A and B, or even arbitrary independent A and B. Dually, while for distinct atomic sentences A, B, we do not expect  $A \vee B$  to be subtractible from A,  $A \vee B$  is instead the formula  $(p \wedge q) \vee (p \wedge \neg q)$ , then the subtraction is available since this amounts to the approved case of subtracting  $p$  from  $p \wedge q$ .<sup>36</sup> It is clear that what we have been calling ultra-independence is the appropriate generalization of the relation between A and B which is here illustrated by having A and B as distinct atomic sentences. For such A and B, the crucial difference between conjunction-to-conjunct and disjunct-to-disjunction inferences is that while  $A \wedge B$

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<sup>36</sup> Thus we are not attempting a revival of Parry’s well known notion of analytic implication. In particular, if A endogenously implies B, and  $s$  is some (uniform) substitution of formulas for variables,  $s(A)$  need not endogenously imply  $s(B)$ .

endogenously implies A (as well as B), A does not (and nor does B) endogenously imply  $A \vee B$ .

It is useful at this point to address a worry about how it can come about that logical implications do not all manage to count as endogenous. As we noted above, the endogenousness consists in an additional requirement which can be put as saying that any part-based subject matter which the implying statement is entirely about, the implied statement is also entirely about. And the worry can be put by asking why this additional requirement isn't automatically satisfied, even with the 'part-based' deleted. Take the case of an arbitrary disjunct-to-disjunction inference. The thought that if A is entirely about  $M$ , then so is  $A \vee B$ , can be made to seem plausible like this: Suppose we had a being who was omniscient in respect of facts entirely about  $M$ , and satisfied the usual rationality assumptions embodied in (normal) epistemic logic; then if A is true and entirely about  $M$ , the being knows that A is true and hence, since  $A \vee B$  follows from A, knows that  $A \vee B$  is true. Now since all we assumed was that the being was (shall we say for short?)  $M$ -omniscient, it must be that the latter disjunction is itself entirely about  $M$ . What is interesting about this admittedly confused worry is the concept in the vicinity of entire aboutness that it draws attention to. Being entirely about  $M$  is an absolute, as opposed to a world-relative concept. But the notions of  $M$ -omniscience and more obviously that of truth ("if A is true...") are world-relative. All that the above line of argument establishes is the triviality that if A is entirely about  $M$ , then  $A \vee B$  follows from something entirely about  $M$ . That does not make the disjunction itself something entirely about  $M$ , because two worlds could agree on all statements entirely about  $M$  and disagree on  $A \vee B$ : since A must take the same truth value in both worlds, being entirely about  $M$ , that truth value must be *False*, or they could not disagree in respect of  $A \vee B$ . But if both worlds do falsify A, they can disagree on the disjunction by disagreeing on B, which of course was not assumed to be entirely about  $M$ . The interesting concept in the vicinity here is the world-relative concept of being settled (in respect of truth value) on the basis of subject matter  $M$ : for brevity, of being  $M$ -settled in  $w$ . A statement is  $M$ -settled in  $w$  iff for all  $u \in W$  such that  $u \equiv_M w$ , A has the same truth-value in  $u$  as it does in  $w$ . Thus if A, still assumed to be entirely about  $M$ , is true in  $w$ , then  $A \vee B$  is  $M$ -settled in  $w$  (and in fact  $M$ -settled as true, in the sense of being true in all  $u$  such that  $u \equiv_M w$ ), whereas if A is false in  $w$  then  $A \wedge B$  (B again arbitrary) is  $M$ -settled in  $w$  (and this time  $M$ -settled as false). The earlier detailing of the worry confused the world-relative notion of being  $M$ -settled with the absolute notion of being entirely about  $M$ . Since the relativity is not idle—what is  $M$ -settled in one world need not be so in another—the notions are not to be conflated. However, they are intimately related, since follows from the definitions that being entirely about  $M$  is equivalent to being, for all  $w \in W$ ,  $M$ -settled in  $w$ .<sup>37</sup> With the relation of endogenous implication defended thus from a worry about collapse into mere logical implication, we conclude our case for the utility of this mereological notion in the theory of logical subtraction.

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<sup>37</sup> These concepts were introduced in §4 of 'A Study in Philosophical Taxonomy'; the content of the last sentence appears as Observation 4.4 there.

## 5. Closing Comments

We have made much of the distinction between orthogonality of subject matters determined by disjoint parts from arbitrary orthogonality of subject matters, in order to show that there is room for a notion stronger than complete logical independence of statements. In discussing parts we have had spatio-temporal (and indeed specifically spatial) parts in mind. But it may be possible to generalize what has been said here about part-based subject matters (and the assignment of regions as the habitats of statements) to other families of subject matters, by discerning a boolean algebraic structure in a domain of ‘generalized parts’. In such a *generalized part-based-like* family of subject matters, the disjointness of any pair of whose elements would be necessary and sufficient for the subject matters determined by them (using the analogue of our function  $\sigma$ ) to be orthogonal. We shall not make any suggestions as to how an account along these lines might proceed in further detail, closing with some remarks aimed at motivating the provision of such an account, along with the concomitant notion of (or analogous to) *habitat*. The important constraint on any such account is to avoid collapsing the distinction, emphasized in our opening section, between (the subject matters determined by ‘generalized’) parts of reality and aspects of reality. Such a collapse would allow any case of independence to be construed in terms of disjoint (generalized) parts, and thus deprive us of such applications of the distinction as were made in the preceding section.<sup>38</sup>

The example alluded to in notes 27 and 28 may serve to illustrate the need to transcend spatial (or temporal) parts without going so far as to count any aspect of how things are as a part of reality. Miller’s example (*op. cit.*), on which our  $L_1/L_2$  discussion was modelled, had a language with three atomic sentences:  $h, r, w$ , for reporting the state of the weather as hot, rainy, windy, respectively, and also a second language with three atomic sentences  $h, m, a$ , for reporting the weather in some fixed spatial region (at some fixed time) as, respectively hot (again), ‘Minnesotan’ and ‘Arizonan’, where the weather is understood to be Minnesotan just in case it is hot if and only if it is rainy, and to be Arizonan just in case it is hot if and only if it is windy. (The ‘if and only if’s here are to be interpreted as material biconditionals.) Then on the assumption that the  $h, r, w$  of the first language are (completely) logically independent, so are the  $h, m, a$  of the second language, and the languages are intertranslatable in a manner which will be apparent from our  $L_1/L_2$  discussion. Miller’s reason for giving this example was to show the unwanted language-dependence of a method (proposed by Pavel Tichý) for basing judgements of comparative verisimilitude on numerical comparisons of the truth and falsity of atomic sentences—which the example shows to yield different results depending on which of the two languages one is working in, even when everything else is held fixed. Our interest is somewhat different. As in Section 3 above, the question arises as to whether we can justify saying that  $(h \wedge r) - h$  should be equivalent to  $r$ . That is how matters look from the perspective of the  $h,r,w$ -language; but when we go over to

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<sup>38</sup> Thus we should in the present context be especially careful to avoid such talk as Peter Unger’s, of ‘parts of the whole truth about everything’ which are claimed (*inter alia*) to be the proper objects of knowledge in his paper, ‘Truth’, pp.257–291 in M. K. Munitz and P. K. Unger (eds.), *Semantics and Philosophy*, New York University Press, New York 1974. (This material also appears in Chapter VII of Unger, *Ignorance*, Clarendon Press, Oxford 1975.)

the  $h, m, a$ -language, we find it has an equivalent to  $h \wedge r$  in the shape of  $h \wedge m$ , so that the corresponding subtraction,  $(h \wedge m) - h$ , will yield the result  $m$ , which is not equivalent to  $r$  (but to  $h \leftrightarrow r$ ). And our mereological approach of Section 4 is this time unable to adjudicate between the two languages in the way we saw to be possible for  $L_1$  and  $L_2$ , since now only a single region is at issue. Thus some generalized part-based-like family of subject matters is called for, if at most one of Miller's two languages is deemed legitimate in the sense of having the logical independence of its atomic formulas reflect the orthogonality of subject matters determined by disjoint generalized parts. Since we are dealing here with logically independent intrinsic properties of a spatial region, it may be that what is called for is some 'dimensional' account of properties along the lines explored by Stalnaker, Sanford, and Gärdenfors,<sup>39</sup> and indeed gestured at by Oddie for application to this very issue.<sup>40</sup>

For an example with a rather different flavour, consider views to the effect that having a particular emotion is a matter of being in a certain affective state and at the same time having certain propositional attitudes. Philippa Foot once remarked on the relevant propositional attitudes in the case of pride—roughly that one is somehow favourably associated with (e.g. by being responsible for) the object of one's pride—and said that in their absence “there can be no pride, not because no one could psychologically speaking feel pride in such a case, but because whatever he did feel could not logically be pride.”<sup>41</sup> To resolve an indeterminacy this remark leaves, let us take the suggestion that pride itself is a kind of (quasi-mereological) composite of affective state and propositional attitude, rather than the affective state itself (on condition that it is accompanied by the relevant attitude). Then if—speaking rather roughly in the interests of brevity— $p$  ascribes pride to a certain individual, and  $q$  ascribes the relevant propositional attitude, it ought to be possible to ascribe the 'phenomenal residue' in question by logically subtracting  $q$  from  $p$ . We do not want to be left, after the subtraction, holding the pseudo-state of being thus phenomenally affected iff appropriately attitudinizing.

The need to avoid the collapse mentioned at the end of the last paragraph but two above suggests that a purely algebraic characterization is unlikely to be of assistance in making good this notion of a generalized part-based-like family of subject matters. For example, if we were to interchange '3' and '4' in the labels of Figure 1 but leave the solid circles where they are so as to mark the new 'pseudo-parts', we should have a boolean sublattice of such pseudo-parts dual-isomorphically embedded in the lattice of partitions exactly as the genuine spatial parts are in Figure 1. A way of generalizing spatio-temporal parthood which excludes such maneuvers while at the same time

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<sup>39</sup> R. Stalnaker, 'Anti-Essentialism', pp.343–355 in P. A. French, T. E. Uehling and H. K. Wettstein (eds.) *Midwest Studies in Philosophy, Vol. IV: Studies in Metaphysics*, University of Minnesota Press, Minneapolis 1979; D. H. Sanford, 'Independent Predicates', *American Philosophical Quarterly* **18** (1981), 171–174; P. Gärdenfors, 'Frameworks for Properties: Possible Worlds vs. Conceptual Spaces', pp.383–407 in L. Haaparanta, M. Kusch, and I. Niiniluoto (eds.), *Language, Knowledge and Intentionality: Perspectives on the Philosophy of Jaakko Hintikka (Acta Philosophica Fennica 49)* 1990.

<sup>40</sup> §6.6 of Graham Oddie, *Likeness to Truth*, Reidel, Dordrecht 1986.

<sup>41</sup> From p.86 of 'Moral Beliefs', pp.83–100 in P. Foot (ed.), *Theories of Ethics*, Oxford University Press, London 1967.

allowing for the kind of extension to non-spatio-temporal cases illustrated in the preceding paragraphs remains to be found.<sup>42</sup>

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<sup>42</sup> I am indebted to audiences in seminars at the Australian National University and Monash University for several suggestions and corrections which have been incorporated into the present version. I am also deeply grateful to David Lewis for his enthusiastic interest in the material on logical subtraction, as it has undergone successive modifications over the years (see note 20). Lewis himself considers what seems to be a closely related issue in his paper 'A Problem about Permission', pp.163–175 in E. Saarinen *et al.* (eds.), *Essays in Honour of Jaakko Hintikka*, Reidel, Dordrecht 1979.