

EQUIVALENTIAL INTERPOLATION

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1. Introduction

By a *consequence relation* on a set L of formulas we understand a relation $\vdash \subseteq \wp(L) \times L$ satisfying the conditions called ‘Overlap’, ‘Dilution’, and ‘Cut for Sets’ at p.15 of [25]; we do not repeat the conditions here since we are simply fixing notation and the concept of a consequence relation is well known in any case. (The characterization in [25] amounts to that familiar from Tarski’s work, except that there is no ‘finitariness’ restriction to the effect that when $\Gamma \vdash A$, for $\Gamma \subseteq L$, $A \in L$, we must have $\Gamma_0 \vdash A$ for some *finite* $\Gamma_0 \subseteq \Gamma$. The presence or absence of this condition makes no difference to anything that follows.) Each language L to be considered will be a sentential language whose formulas are built in the usual way by application of k -ary (primitive) connectives to k simpler formulas, starting with the simplest formulas – the propositional variables (or ‘sentence letters’) – not constructed with the aid of connectives. We assume, as usual, that there are countably many such variables, and they will be denoted by p, q, r, \dots possibly with numerical subscripts. A consequence relation \vdash on such an L has the *Unrestricted Interpolation Property* when for any $A, C \in L$ with $A \vdash C$, there exists $B \in L$ with $A \vdash B$ and $B \vdash C$, such that C is constructed only out of such propositional variables as occur both in A and in C . (Such a B is called an interpolant for A and C .) Note that we take the usual notational liberties here, writing “ $A \vdash C$ ” (and the like) for “ $\{A\} \vdash C$ ”, “ $\Gamma, A \vdash C$ ” to mean “ $\Gamma \cup \{A\} \vdash C$ ”, and “ $\vdash C$ ” to mean “ $\emptyset \vdash C$ ”. Further, we sometimes abbreviate the claim that $A \vdash B$ and $B \vdash C$ to “ $A \vdash B \vdash C$ ”, and when C is A itself, we always write this simply as “ $A \dashv\vdash B$ ”. Finally, we shall also write “ $A \dashv\vdash B \dashv\vdash C$ ” to mean: $A \dashv\vdash B$ and $B \dashv\vdash C$ (equivalently: $A \vdash B \vdash C$ and $C \vdash B \vdash A$).

The above reference to k -ary connectives subsumes the case of $k = 0$, such a connective being already a formula (a ‘sentential constant’) in which no propositional variables occur. If L has no such constants, then it is in general not possible for \vdash to have the unrestricted interpolation property since as long as we have $A \vdash C$ for $A, C \in L$ with no propositional variables in common, there are no candidate formulas to play the role of the interpolant B , as every formula in a sentential language without constants must contain at least one variable and this will have to be a variable foreign to A or else to C . For this reason, we define \vdash to have the (*Restricted*) *Interpolation Property* – henceforth IP – just in case for all formulas $A, C \in L$ such that A and C have at least one propositional variable in common, if $A \vdash C$ then there exists $B \in L$ with $A \vdash B$ and $B \vdash C$, with B constructed only out of such variables as are common to A and C . In other words, this is as for the unrestricted version of the property, except that we do not demand an interpolant B for “variable-disjoint” A and C for which $A \vdash C$. (An alternative response, which

for some but not all choices of \vdash has the same upshot, is to demand an interpolant for A and C in every case except when either $A \vdash D$ for all D or else $\vdash C$.) This relaxation will allow us to consider in particular the equivalential—or ‘biconditional’—fragments of intuitionistic and classical logic, in which the only connective with which formulas are constructed is the binary connective \leftrightarrow as well as other constant-less fragments, providing us with one of the senses of our title: equivalential interpolation as IP for such fragments. (IP will also be considered for other fragments.) What we shall actually call the Equivalential Interpolation Property in §4 below, however, takes the adjective “equivalential” in another sense, as pertaining not to any equivalence *connective* but to the *relation* of logical equivalence (mutual consequence). As we shall see, for the case of classical logic, there is an interesting connection – in the case of classical logic – between ordinary interpolation (IP, that is) for the equivalential fragment and this bilateral version of IP. In fact, we shall derive the former from the latter. The result itself is not new, since it is known that every fragment of classical logic enjoys IP. ([4] and [13] give versions of approximately this result – though see §5 below.) But this way of deriving it does perhaps have a certain interest in its own right.

“Equivalential” in our title is thus ambiguous, meaning on the one hand “pertaining to $\dashv\vdash$ ” and, on the other, “pertaining to \leftrightarrow ”. Since the way these two senses are related in §4 makes heavy use of the effect of (uniform) substitution, on the side of the Equivalential Interpolation Property, we consider in §2 the extent to which the interpolants promised by IP itself can be guaranteed to exist by judicious substitutions. In §3 we discuss IP for the equivalential fragment of intuitionistic logic and also for fragments of some intermediate logics, one of which throws light on an example employed in §2. Some remaining questions are collected together and raised in §5.

There is a considerable literature on IP for propositional logic, and related conditions, only a small portion of which will be cited in what follows. It seems therefore appropriate to mention that Chapter 14 of [1] ends with an up-to-date guide to this literature. Chapter 2 of [1] gives the Kripke semantics for intuitionistic and intermediate logics; the latter topic will occupy us in §3. A complete overview of the subject may be obtained by consulting [18]; for a briefer treatment, see the survey [6]. As well as dealing with modal logics, the body of Chapter 14 of [1] also treats interpolation for intermediate logics. (In fact the authors, as for several other of our bibliographical references, use the terminology of superintuitionistic logics. To translate, recall that in the case of propositional logic – though not predicate logic – the superintuitionistic logics are precisely the intermediate logics together with inconsistent logic.)

2. Substitution and Interpolation

Our introduction passed, without explicit notice, from talk of consequence relations with the IP to talk of logics (and fragments of logics) with this property. Frequently, for reasons we need not go into, the idea of viewing a consequence relation as a logic is associated with the further demand that to count as a logic, a consequence relation should be “closed under Uniform Substitution”, or, as we shall say, be *substitution-invariant* meaning that \vdash should satisfy, for any substitution (of formulas for propositional variables) s , the condition that whenever $\Gamma \vdash A$, we must also have $s(\Gamma) \vdash s(A)$. Here $s(\Gamma)$ is just $\{s(C) \mid C \in \Gamma\}$. (Substitution-invariant consequence relations are sometimes called *structural* consequence relations in the literature.)

Whatever the merits of this condition as a mark of ‘logicality’, it has some ramifications for the issue of IP.

Let us imagine what the simplest possible case of a failure of IP might look like, for a consequence relation \vdash assumed substitution-invariant. We want A, C , with $A \vdash C$, where A and C share some propositional variable, p , say, but there being no interpolant B (with $A \vdash B \vdash C$). A must contain, in addition to p , some other variable not occurring in C , or else A itself could serve as B , and since we are after the simplest possible case, let us suppose that there is just one such additional variable in A : q , say. Likewise, we need to suppose that C contains, in addition to p , some other variable – r , say – not occurring in A ; otherwise C itself could play the role of B . Thus the simplest counterexample to IP for a yet to be specified \vdash would take A as a compound of p with q , constructed using some binary connective $\#$ in the language of \vdash , and similarly C would be a compound of p with r , constructed using a binary connective \star , and we had (1):

$$p \# q \vdash p \star r \tag{1}$$

So far, in our quest for a simple counterexample to IP, we have taken simplicity to be a matter of minimizing the number of distinct occurrences of connectives and sentence letters in the example. Suppose we go further and reduce the number of connectives occurring as well as the number of occurrences of connectives, insisting for the case of (1), in other words, that \star is $\#$. Fixing on A as $p \# q$, this allows two possibilities for C : as $p \# r$ and as $r \# p$. An example as simple as this of failure of IP cannot, we now observe, arise for substitution-invariant \vdash . Let us take the first possibility for C first; that is, we are given, for substitution-invariant \vdash :

$$p \# q \vdash p \# r \tag{2}$$

Substituting p for q in (2) gives (3), while substituting p for r in (2) gives (4):

$$p \# p \vdash p \# r \tag{3}$$

$$p \# q \vdash p \# p \tag{4}$$

From (3) and (4), we conclude that $p \# p$ serves as an interpolant for the chosen A and C , since we have just established

$$p \# q \vdash p \# p \vdash p \# r \tag{5}$$

If on the other hand – the second of the two possibilities distinguished above – C is $r \# p$, then corresponding to (2) we have:

$$p \# q \vdash r \# p \tag{6}$$

and appeals to the substitution-invariance of \vdash yield (7) and (4) again, from which we may again draw the (5)-like conclusion (8):

$$p \# p \vdash r \# p \tag{7}$$

$$p \# q \vdash p \# p \vdash r \# p \tag{8}$$

What moral can we draw from the interpolations obtained at (5) and (8) here? Though their derivations make clear the reasoning in what was after all supposed to be the simplest case, the more general point is best brought out by introducing the idea of two formulas having, as we shall say, the same ‘skeleton’. To define this, let us denote by ε_0 the substitution which replaces

every variable by the variable p . Then we say that formulas A and B *have the same skeleton* just in case $s_0(A) = s_0(B)$. Clearly, the same effect would be achieved whichever variable we decided to use to play the role played by p here: if we visualize the construction-tree of a formula as having its non-terminal nodes labelled with connectives and its terminal nodes labelled with propositional variables, then what we describe as formulas with the same skeleton are formulas whose construction trees are identical save in respect of the labelling of the terminal nodes. Note that a substitution-instance of a formula A (i.e., the formula $s(A)$ for some substitution s) will not in general have the same skeleton as A , since the substitution in question may not be a variable-for-variable substitution, and that formulas A and B having the same skeleton will not in general be, either of them, a substitution-instance of the other; this is illustrated, for example, by the case of the formulas (of the language of intuitionistic or classical sentential logic) $p \wedge (q \vee (p \rightarrow r))$ and $p \wedge (p \vee (q \rightarrow r))$. The important thing about the formulas on the left and right of the \vdash in (2) and (6), which allowed the appeal to substitution-invariance to supply interpolants, was that in each case the formulas involved had the same skeleton. It was not important that these formulas were constructed by a single application of a primitive connective, for instance – though if they are, it is important that this be the *same* connective (or they would not have the same skeleton), and it is not important that there is only one variable in A but not in C or in C but not in A . We distil the general pattern from the above examples as:

PROPOSITION 2.1 *Suppose that \vdash is a substitution-invariant consequence relation in the language of which there are formulas A, C , having the same skeleton and exactly one variable in common, and for which $A \vdash C$. Then there is a formula B in which that common variable is the only variable to occur, with $A \vdash B \vdash C$. Furthermore, B has the same skeleton as A and C .*

Proof. Suppose A and C meet the conditions given, Since we have an infinite supply of variables and \vdash is substitution-invariant, there is no loss in generality in assuming that the one variable common to these two formulas is p . Let the remaining variables in A but not C be q_1, \dots, q_m and the those variables in C which are not in A be r_1, \dots, r_n . Thus fully indicating the propositional variables involved, we have $A = A(p, q_1, \dots, q_m)$, $C = C(p, r_1, \dots, r_n)$, and our task is to find $B = B(p)$ with $A \vdash B \vdash C$. Let s_1 be the substitution mapping all of q_1, \dots, q_m to p , and s_2 the substitution mapping all of r_1, \dots, r_n to p . Now, since $A \vdash C$ and \vdash is substitution-invariant we have $s_1(A) \vdash s_1(C)$ and $s_2(A) \vdash s_2(C)$. Since the q_i do not occur in C and the r_j do not occur in A , $s_1(C) = C$ and $s_2(A) = A$. Thus we have (i) $s_1(A) \vdash C$ and (ii) $A \vdash s_2(C)$. Recalling the substitution s_0 from the definition of what it is for formulas to have the same skeleton, note that since A and C are supposed to stand in this relation, and $s_0(A) = s_1(A)$, $s_0(C) = s_2(C)$, we have $s_1(A) = s_2(C)$. Thus (i) and (ii) gives us $s_1(A)$ (*alias* $s_2(C)$) as the promised interpolant B , in which only the common variable p appears. \square

We may ask how essential the restriction is to there being a *single* propositional variable common to A and C here. Certainly the above method of proof does not deliver an interpolant in the several common variables, should there be more than one such. For example, let the language of \vdash have two binary connectives $\#$ and \star (as in (1)), and suppose

$$(p_1 \# q) \star p_2 \vdash (p_2 \# p_1) \star r \tag{9}$$

We have a choice between p_1 and p_2 as the variable to play the role of p in the above derivation of (5) or (8), or more generally in Proposition 2.1, where the formulas on the left and the right of the ‘ \vdash ’ in (9) are the A and C between which we wish to interpolate a suitable B. Suppose we pick p_1 as the variable to replace r with on the left, giving (10):

$$(p_1 \# q) \star p_2 \vdash (p_2 \# p_1) \star p_1 \tag{10}$$

We are now at the $A \vdash_{\mathfrak{s}}(C)$ stage of the proceedings, in need of $\mathfrak{s}(A)$ for which $\mathfrak{s}(A) \vdash C$ and $\mathfrak{s}(A) = \mathfrak{s}(C)$, so that this formula will serve as B. For the latter equality, we need $\mathfrak{s}(p_1) = p_2$, $\mathfrak{s}(q) = p_1$, $\mathfrak{s}(p_2) = p_1$. So to have $\mathfrak{s}(A) \vdash C$, we need (11):

$$(p_2 \# p_1) \star p_1 \vdash (p_2 \# p_1) \star r \tag{11}$$

But (11) does not follow from (9) and the substitution-invariance of \vdash , because the required substitution disrupts the right-hand side (which should instead, applying \mathfrak{s} as described above, be the formula $(p_1 \# p_2) \star r$). A similar difficulty blocks the alternative route of selecting p_2 as the variable with which to replace r on the left of (9). We shall not enquire as to what suitable generalization of the same-skeleton relation might fix this problem for allowing a more liberal version of Proposition 2.1, with several common variables—perhaps one relativized to the set of common variables and requiring them to remain in position as opposed to flitting around as in the A and C of (9), since the result is of course very limited when our interest is establishing IP itself for a particular \vdash .

Accordingly, to close this section, we return to the simplest potential counterexample to IP in which the A and C concerned do not have the same skeleton, namely (1) above, repeated for convenience here:

$$p \# q \vdash p \star r \tag{1}$$

To realize the potential, let us give a matrix semantics for some \vdash with (1) satisfied but with no interpolant available. As explained in §1, any such matrix must have at least three elements, and to this end we give the following tables describing a three-element algebra with two fundamental operations as indicated:

#	1	2	3
1	1	1	1
2	2	2	1
3	3	3	3

☆	1	2	3
1	1	1	1
2	1	1	1
3	1	2	1

Figure 1

To convert this algebra into a (logical) matrix, we must supply a set of designated values. We stipulate that for the matrix we are interested in, the value 1 is the (sole) designated value: in other words, we define \vdash on the language whose only connectives are # and ☆: for any set of formulas Γ of this language and formula C, $\Gamma \vdash C$ if and only every homomorphism h from the language (considered as an algebra with the set of formulas as its universe and the primitive connectives as fundamental operations) to the above algebra, for which $h(A) = 1$ for every $A \in \Gamma$ is also such that $h(C) = 1$. Another way of putting this is to say that \vdash is the consequence relation *determined* by the matrix we have specified. (A precise general definition appears in the passage marked ‘Digression’ before Lemma 4.2 below.) We give the present example in a self-contained way here, with the background supplied in the following section.

PROPOSITION 2.2 *For the consequence relation \vdash determined by the three-element matrix described, (1) holds but there is no formula $B(p)$ for which $p \# q \vdash B(p) \vdash p \star r$*

Proof. To check that (1) holds, suppose that for some homomorphism h , we have $h(p \star r) \neq 1$. From the table for \star we see that in that case $h(p \star r) = 2$, and $h(p) = 3$, (and $h(r) = 2$, though we do not need this fact). But from the $\#$ -table, we see that since $h(p) = 3$, $h(p \# r) = 3$ regardless of $h(r)$. In particular, then, from the hypothesis that $h(p \star r) \neq 1$, we have been able to conclude that $h(p \# r) \neq 1$, confirming (1). For the non-existence of an interpolant, one can verify by induction on the construction of $B = B(p)$ that any such formula falls into one of two types: either (i) for all h , $h(B) = 1$, or (ii) for all h , $h(B) = h(p)$. If a given B is of type (i), we do not have $B \vdash p \star r$, since we can put $h(p) = 3$, $h(r) = 2$, giving $h(p \star r) = 2$ while $h(B) = 1$, and if B is of type (ii), we do not have $p \# q \vdash B$, since with $h(p) = 2$ and $h(q) = 3$, we get $h(p \# q) = 1$ while $h(B) = 2$. \square

3. Interlude: Remarks on Two Examples

As mentioned in our introduction, we can consider IP for the equivalential (\leftrightarrow -based, that is) fragments of various logics of interest, and we can also consider, for whatever fragment, an equivalential version (equivalence in the sense of the relation $\dashv\vdash$ rather than the connective \leftrightarrow) of IP; in the following section we shall connect the two topics in the case of classical logic. Here we offer a few remarks on the first of them as it applies to intuitionistic logic, and also touch on IP for the implicative fragment some intermediate logics. The two examples mentioned in our title, from these respective areas, appear as (16) and (20) in what follows.

Renardel de Lavalette, having found failures of IP for uncountably many fragments of intuitionistic logic in [23], turned his attention in [24] to what he called ‘elementary’ fragments; these papers may be consulted for precise definitions, but the general idea is that for an arbitrary fragment any connective definable in terms of the familiar primitives \wedge , \vee , \rightarrow , and \neg (or \perp) may be taken as a new primitive, while the *elementary* fragments are obtained by making any selection from this list of familiar connectives. IP was established for all such elementary fragments, and §6 of [24] discusses some not-quite-elementary fragments of which the \leftrightarrow -fragment is the most interesting: although unable to adapt other ways of showing IP to yield the result for this fragment, Renardel de Lavalette concludes his discussion of the difficulties with the conjecture that this fragment does indeed satisfy IP.

A more pessimistic evaluation of the prospects of IP for the fragment in question is suggested by a perusal of [20], however. Porębska there investigates a property analogous to IP but for equational logic, possession of which by the quasi-equational theory of a class \mathbf{K} of algebras of a fixed similarity type implies the following, where the ε_i are equations for the type in question:

If $\mathbf{K} \models \varepsilon_1 \Rightarrow \varepsilon_3$ and ε_1 and ε_3 have at least one (individual) variable in common, then there is an (“interpolant”) equation ε_2 containing only variables common to ε_1 and ε_3 , for which we have:

$$\mathbf{K} \models \varepsilon_1 \Rightarrow \varepsilon_2 \qquad \text{and} \qquad \mathbf{K} \models \varepsilon_2 \Rightarrow \varepsilon_3 \qquad (12)$$

(Here $\mathbf{K} \models \varepsilon_i \Rightarrow \varepsilon_j$ means that for each $\mathbf{A} \in \mathbf{K}$, $\mathbf{A} \models \varepsilon_i \Rightarrow \varepsilon_j$, i.e., $\varepsilon_i \Rightarrow \varepsilon_j$ is a quasi-identity of \mathbf{A} .) In fact, Porębska considers a more general property, allowing a set of equations instead of just a

single equation in the ε_1 -position here, but since [20] disproves the general interpolation property for the particular \mathbf{K} we are interested in *via* a counterexample in which there is only a single equation in that position, this special case will suffice for the present recapitulation. (The more general equational interpolation property is also considered in [27] and [15], both of which discuss earlier literature. [15] calls this property IPE, and [27] calls it EIP, though this what we call EIP in the following section is a different matter – an *equivalential* as opposed to an *equational* interpolation property.) The \mathbf{K} in question is the class (indeed, variety) of *equivalential algebras* introduced in [11] as an algebraization of the equivalential fragment of intuitionistic logic, and Porębska presents (proof of Lemma 1, [20], p.36) as an example of $\varepsilon_1, \varepsilon_3$, for which $\varepsilon_1 \Rightarrow \varepsilon_3$ holds in every such algebra even though there is no ε_3 to be found (in the shared variables) satisfying (12), something we quote in (13), explaining the notation afterwards:

$$\mathbf{E} \models a = bcc \Rightarrow daadb = b \quad (13)$$

Here \mathbf{E} is the class of equivalential algebras, and ‘ a ’, ‘ b ’,... are individual variables. The operation symbol – ‘ \leftrightarrow ’, say, since there is no danger of confusion with the connectival use of this symbol – is suppressed, and then parentheses are dropped in favour of association to the left. Restoring the missing notation, then, ε_1 is $a = (b \leftrightarrow c) \leftrightarrow c$, and ε_3 is $((d \leftrightarrow a) \leftrightarrow a) \leftrightarrow d \leftrightarrow b = b$, and according to [20] there is no ε_2 in which the only variables to appear are ‘ a ’ and ‘ b ’, satisfying (12) for $\mathbf{K} = \mathbf{E}$. The defining equational axioms for equivalential algebras (given in [11]) suffice for a proof of $t \leftrightarrow t = u \leftrightarrow u$, for any terms t and u , we can introduce a constant $\mathbf{1}$ for this fixed element of any such algebra, and in fact ([11], p.420), since for any $\mathbf{A} \in \mathbf{E}$, we have, for any terms t, u , $\mathbf{A} \models t = u \Leftrightarrow tu = \mathbf{1}$, we can rewrite (13) as:

$$\mathbf{E} \models a(bcc) = \mathbf{1} \Rightarrow daadb = \mathbf{1} \quad (14)$$

If Porębska’s argument for the ‘uninterpolability’ of (13) is correct, then the same should hold for (14), i.e., there should exist no term t whose only variables are ‘ a ’ and ‘ b ’, for which (15) is satisfied:

$$\mathbf{E} \models a(bcc) = \mathbf{1} \Rightarrow t = \mathbf{1} \quad \text{and} \quad \mathbf{E} \models t = \mathbf{1} \Rightarrow daadb = \mathbf{1} \quad (15)$$

In view of the relationship between the variety \mathbf{E} and the equivalential fragment of intuitionistic logic ([11], p.427), this it is hard to see this as meaning anything other than that IP fails for this fragment, with

$$p_1 \leftrightarrow ((p_2 \leftrightarrow q) \leftrightarrow q) \vdash (((r \leftrightarrow p_1) \leftrightarrow p_1) \leftrightarrow r) \leftrightarrow p_2 \leftrightarrow p_2 \quad (16)$$

as a counterexample based on (13). In replacing individual variables by propositional variables here (as well as restoring the ‘ \leftrightarrow ’ and parentheses) we have followed the convention of §2 and reserved p, q, r , with or without subscripts, for variables occurring, respectively, in both A and C, in A but not C, and in C but not A, of the formulas A, C for which we are given $A \vdash C$ and are wondering whether there is—to take the case in which there is only one variable in the first (“shared”) category— $B(p)$ with $A \vdash B(p) \vdash C$. In the case of (16), of course, there are two variables in this shared category and we should want, for IP, $B(p_1, p_2)$ as a consequence of the left-hand formula in (16) and having the right-hand formula in (16) as a consequence. Unless there is a mistake in [20] or in the above sketch of a transfer from the reasoning about quasi-identities in equational logic to consequence statements in sentential logic, no such $B(p_1, p_2)$ can be found and, contrary to Renardel de Lavalette’s conjecture, IP fails for the equivalential

fragment of intuitionistic logic. (*Personal note:* The qualified nature of this conclusion is not meant to cast aspersions on the relevant proof in [20] – though I confess to not having been able to follow it. Given a longer period than remains available for the timely completion of this contribution, a routine though laborious check through the 9 pairwise non-equivalent \leftrightarrow -formulas in two variables, conveniently enumerated at the end of [11], would reveal the existence or—as the above reasoning leads us to expect—the non-existence of a suitable interpolant for (16).)

If we allowed ourselves to step outside the equivalential fragment for a moment, we could easily come up with an interpolant for (16): $p_2 \rightarrow p_1$. This is probably most easily seen if we allow ourselves to rewrite (16) by helping ourselves to \rightarrow when convenient:

$$p_1 \leftrightarrow ((p_2 \rightarrow q) \rightarrow p_2) \vdash (((r \rightarrow p_1) \rightarrow r) \rightarrow r) \rightarrow p_2 \rightarrow (((r \rightarrow p_1) \rightarrow r) \rightarrow r) \quad (17)$$

To convert (16) into (17), we have made use of the intuitionistic equivalence, for arbitrary formulas D, E , of $(D \leftrightarrow E) \leftrightarrow E$ with $(D \rightarrow E) \rightarrow D$ and of $((D \leftrightarrow E) \leftrightarrow E) \leftrightarrow D$ with Peirce’s Law in the form $((D \rightarrow E) \rightarrow D) \rightarrow D$ (which results from the former equivalence when the redundant converse implication is discarded). Our interpolant $p_2 \rightarrow p_1$ is an obvious intuitionistic consequence of the formula on the left of (17) – in fact of just the ‘ \leftarrow ’ half of that formula. To see, informally, that it has the formula on the right of (17) as an intuitionistic consequence assume the latter formula’s antecedent: $((r \rightarrow p_1) \rightarrow r) \rightarrow r$. Together with $p_2 \rightarrow p_1$, this gives $((r \rightarrow p_1) \rightarrow r) \rightarrow p_1$. But p_1 is an intuitionistic consequence of this formula—a fact sometimes known as Wajsberg’s Law (a constructively acceptable weakening of Peirce’s Law; see [2], p.263, or – for the original source – [26], p.151, Example 4); and the consequent of the right-hand formula in (17) follows easily from p_1 . A further advantage in considering implicational reformulations (when they exist) of equivalential formulas is the greater ease of checking what follows intuitionistically from what with the aid of Kripke models, which is a cumbersome procedure with \leftrightarrow -formulas because of the need to keep constant track of the two ways for such formulas to fail to be true at a point.

The second example to be discussed here is also (a simplification of) an example considered by Porębska – this time, in [21] – and it also has a conspicuously ‘Peircean’ flavour. Again we are concerned with fragments, though here it is fragments of a range of intermediate logics rather than just of intuitionistic logic itself. Lemma 5 of [21] cites the following example in the course of showing that, as Porębska puts it, the “intermediate logic determined by [the] 4-element chain fails to have the IP for every fragment containing implication”:

$$((p \rightarrow (((q \rightarrow r) \rightarrow q) \rightarrow q)) \rightarrow p) \rightarrow (((s \rightarrow p) \rightarrow s) \rightarrow s) \quad (18)$$

Since we have been taken IP to pertain to consequence-statements rather than to (provable) implicational formulas, we should think of the example with \vdash replacing the main \rightarrow , and to conform with our conventions on variable choice (variously decorated ‘ q ’ for those on the left but not on the right, etc.), we rewrite (18) as (19). (See further §5 on this point.) In the further interests of readability, we have put ‘ $A \rightarrow B \rightarrow C$ ’ in place of ‘ $(A \rightarrow B) \rightarrow C$ ’ for certain subformulas of the former form:

$$(p \rightarrow ((q_1 \rightarrow q_2 \rightarrow q_1) \rightarrow q_1)) \rightarrow p \vdash (r \rightarrow p \rightarrow r) \rightarrow r \quad (19)$$

The reference to the four-element chain in the quotation from Porębska above is to the linearly ordered Heyting algebra with four elements; in terms of the Kripke semantics, this is the logic determined by the three-element chain (the extra element in the associated algebra of hereditary subsets in this case corresponding to \emptyset), or equivalently, by the class of linear frames in which there are no chains of four (distinct) elements. [21] may be consulted for a proof of the absence of an interpolant – using any connectives (not just \rightarrow) – in the logic concerned for (19). (For classical logic, we could just use $p \rightarrow p$.) This is no surprise in view of the negative results on IP for intermediate logics uncovered by Komori [12], Zachorowski [28], and Maksimova [15]; the logic lies squarely within the range for which these papers showed IP to be lacking. In the case of [15], not only necessary, but also sufficient, conditions for the failure of IP are given, culminating in the dramatic result that of the continuum many intermediate logics, precisely seven have IP. (We are speaking of intermediate *propositional* logics of course. The – very different – situation for intermediate predicate logics is reported on in [17].) These are, aside from intuitionistic and classical logic, the Dummett-Lemmon intermediate logics KC and LC, and the logic with what is sometimes called Jankov’s axiom(-schema) $A \vee (A \rightarrow (B \vee \neg B))$ along with two of this latter logic’s extensions. An aside: as this description of the logics makes evident—and as Maksimova notes explicitly—this means that intuitionistic logic itself is the only intermediate logic with both the Interpolation Property and the Disjunction Property, the latter being possessed by those \vdash for which $\vdash A \vee B$ only when $\vdash A$ or $\vdash B$. (There is a purely implicational axiom due to S. Nagata, the second in a countable sequence of weakenings of Peirce’s Law – see [6], or p.14 or entry 102 in [18] – which gives the same logic when added to full intuitionistic propositional logic as Jankov’s axiom, though its use makes the absence of the Disjunction Property less immediately obvious.)

The uninterpolability example (with $\#$, \star) of Proposition 2.2 above in fact derives from a simplified version of Porębska’s (19), namely (20):

$$(p \rightarrow q) \rightarrow p \vdash (r \rightarrow p \cdot \rightarrow r) \rightarrow r \tag{20}$$

(20) holds for the consequence relation determined by the class of all (Kripke models based on) frames in which there are no chains of length 3, the logic of which is given by (intuitionistic logic plus) Jankov’s schema, mentioned above. Since, as noted, this logic is one of the “Maksimova seven”, to use (20) as an uninterpolability example, we need to restrict attention to the implicational fragment of the logic. Since a would-be interpolant would have to contain no variables other than p , and the one-variable fragments of all intermediate logics coincide with that of classical logic, the only candidates (to within equivalence) are p and $p \rightarrow p$. Since the former candidate fails to follow from the left-hand formula of (20) and the latter (as well as the former) fails to have the right-hand formula as a consequence, there is no interpolant in the fragment concerned. Stepping outside the fragment, we can use $\neg\neg p$, since this follows (even intuitionistically) from the left-hand formula, and it has the right-hand formula as a consequence when attention is restricted to the frames without 3-chains. One such frame consists of two elements, x_0 and x_1 , the latter being accessible to the former but not conversely. The hereditary subsets are then $\{x_0, x_1\}$, $\{x_1\}$ and \emptyset , and for brevity let us rebaptize them as 1, 2 and 3, respectively. A three-valued table will then record the sets of points at which an implicational formula is true in an arbitrary model on this frame, given the truth-sets for the antecedent and consequent. For instance, if A has the value 2 and B has the value 3 then the $A \rightarrow B$ is true at $2 \rightarrow 3$, which should be 3, since $A \rightarrow B$ isn’t true at either x_0 or x_1 , in this case, each of these points having accessible to it a point (namely x_1) at which A is true and B is not. The double use of \rightarrow here

will cause no confusion; in “ $2 \rightarrow 3$ ” it stands for the operation of relative pseudo-complementation in a three element Heyting algebra whose \rightarrow reduct we have just been considering. This is the algebra of what is sometimes called Gödel’s three-valued logic, which is the logic determined by the matrix that results when 1 is taken as the designated value. (Historical and other details may be found at numerous points – the index should be consulted – in [25]; p.251 there depicts the three-element \rightarrow table, usefully contrasted with those of Kleene and Łukasiewicz, though the authors write ‘0’, ‘ $\frac{1}{2}$ ’, and ‘1’ for the present ‘1’, ‘2’, and ‘3’.) If, instead of taking them as primitive, we introduce the binary connectives $\#$ and \star of §2 into this setting by the following definitions:

$$A \# B = (A \rightarrow B) \rightarrow A \qquad A \star B = ((B \rightarrow A) \rightarrow B) \rightarrow B$$

then we end up with the tables of Figure 1 in that section. Thus the uninterpolability of (20) in the \rightarrow -fragment of the logic determined by the two-element (connected) Kripke frame was redescribed in Proposition 2.2 as an uninterpolability result for what is, in effect, the $\{\#, \star\}$ -fragment. We simply rewrote (20) in terms of $\#$ and \star as (1) ($= p \# q \vdash p \star r$). This gives an alternative proof for Proposition 2.2 to that provided above, since the absence of an implicational interpolant certainly guarantees the absence of one in the $\{\#, \star\}$ -fragment, in view of the above definitions of $\#$ and \star .

Example (20) was in fact used, in a slightly disguised form, to illustrate uninterpolability-for-fragments over twenty years ago, in the final paragraph of [15]. Maksimova was there considering which of the seven intermediate logics with IP also had IP for its positive (i.e. $\{\wedge, \vee, \rightarrow\}$ -based) fragment, and she found that this did not hold in (precisely) the case of the three “Jankov’s schema” logics. Failure of IP for the positive fragments was illustrated by means of the following example (presented here in our notation):

$$(q \rightarrow p) \wedge ((p \rightarrow q) \rightarrow q) \vdash ((p \rightarrow r) \wedge (r \rightarrow p \rightarrow p)) \rightarrow r \tag{21}$$

Once we notice that the pattern, appearing twice here, $(B \rightarrow A) \wedge ((A \rightarrow B) \rightarrow B)$ is intuitionistically equivalent to $(A \rightarrow B) \rightarrow A$, we can see that (21) is just a longwinded way of writing (20).

Finally, to close these remarks with a return to matters equivalential, we should note the implicational formulas in (20) can be rewritten in equivalential terms by using the same intuitionistic equivalences as were mentioned just after (17) above. This turns (20) into:

$$(p \leftrightarrow q) \leftrightarrow q \vdash ((r \leftrightarrow p) \leftrightarrow p) \leftrightarrow r \tag{22}$$

Thus, since the candidate \leftrightarrow -interpolants are (much as before, with \rightarrow) only – to within equivalence – p and $p \leftrightarrow p$, neither of which will serve, we conclude that not only the implicational fragments but also the equivalential fragments of the “2-chains allowed but 3-chains forbidden” intermediate logics—are seen to lack IP. (These are the “intermediate logics on the second slice” in the classification of Hosoi – see for instance [6].)

4. The Equivalential Interpolation Property

We now turn to the bilateralized version of IP, which we shall call the Equivalential Interpolation Property, promising interpolants for A and C , not on the hypothesis that $A \vdash C$, but on the much stronger hypothesis that $A \dashv\vdash C$. This may seem to render talk of interpolation inappropriate,

since there is on this hypothesis no logical distance between A and C for anything to be interpolated into, but we wish to retain the associations of the label – especially because the kind of use made of Uniform Substitution *à propos* of IP in §2 is far more helpful here. We define a consequence relation \vdash to have the *Equivalential Interpolation Property* (or EIP, for short) just in case for any formulas A, C, with at least one variable in common, if $A \dashv\vdash C$, then there is a formula B containing only variables common to A and C, for which we have $A \dashv\vdash B \dashv\vdash C$. Three quick points to note about this definition: (1) we could, by omitting the condition that A and C have at least one variable in common, have defined what by analogy of the terminology in §1 would be called the Unrestricted EIP; (2) the consequent of the definition, to the effect that $A \dashv\vdash B \dashv\vdash C$, could equivalently—given the hypothesis that $A \dashv\vdash C$ —have been expressed as $A \vdash B \vdash C$, to make things look more like the definition of IP in §1; (3) for a given \vdash , EIP follows from IP (because if $A \dashv\vdash C$ then by IP and the fact that $A \vdash C$, we have B with $A \vdash B \vdash C$, whence, since $C \vdash A$, we have $A \dashv\vdash B \dashv\vdash C$).

Now, whereas the mere substitution-invariance of \vdash only secured interpolants as demanded by IP for formulas A, C, satisfying highly specific further conditions (given in Proposition 2.1), this property of \vdash is sufficient without further conditions to provide the equivalential interpolants demanded by EIP in all cases:

PROPOSITION 4.1 *Every substitution-invariant consequence relation has EIP.*

Proof. Suppose \vdash is substitution-invariant, and A and C are formulas with at least one variable in common, for which we have $A \dashv\vdash C$. Let the common variables be p_1, \dots, p_l ($l \geq 1$), any variables in A but not C be q_1, \dots, q_m and any variables in C but not A be r_1, \dots, r_n ($m, n \geq 0$). Our hypothesis thus, with occurring variables made explicit is:

$$A(p_1, \dots, p_l, q_1, \dots, q_m) \dashv\vdash C(p_1, \dots, p_l, r_1, \dots, r_n)$$

Now let s be a substitution which replaces each of q_1, \dots, q_m by p_l . By substitution-invariance, we have $s(A) \dashv\vdash s(C)$. Since the q_i do not occur in C, $s(C) = C$ and thus $s(A) \dashv\vdash C$; recalling that $A \dashv\vdash C$, we conclude that $A \dashv\vdash s(A) \dashv\vdash C$. Thus $s(A)$ can be taken as the desired equivalential interpolant (the B of the definition of EIP). \square

To put this to some use in obtaining IP for the \leftrightarrow -fragment of classical logic, we need first to make it clear what this fragment is. We shall treat it as a consequence relation, to be called \vdash_{\leftrightarrow} , on the language with just one connective \leftrightarrow , that connective being binary, and it will simply be that given by the familiar truth-table account of \leftrightarrow . It will help to put this account in terms of the determination of a consequence relation by a class of valuations. By a valuation (for a given language) is here meant any function from the set of formulas (of that language) to the two-element set $\{T, F\}$, and we say that a consequence relation \vdash is *determined* by a class \mathcal{V} of valuations (for the language of \vdash) just in case for all formulas C and sets of formulas Γ , we have $\Gamma \vdash C$ if and only if for every $v \in \mathcal{V}$, if $v(A) = T$ for each $A \in \Gamma$, then $v(C) = T$. Finally we call a valuation v *\leftrightarrow -boolean* when for all formulas A and B, $v(A \leftrightarrow B) = T$ if and only if $v(A) = v(B)$. This allows us to define \vdash_{\leftrightarrow} as the consequence relation (on the language described) determined by the class of all \leftrightarrow -boolean valuations (for that language), and also to make a more general observation. For the latter, we introduce an operation on pairs of valuations which is analogous to the truth-function we have (in effect) just associated with \leftrightarrow . (More precisely, we could

describe this operation as the operation Galois-dual to \leftrightarrow in terms of a Galois connexion detailed in [9], but we do not need the general apparatus for this particular application.) We denote this operation by “ \sim ”, defining, for valuations u, v , the valuation $u \sim v$ to be the unique valuation w such that for all formulas A , $w(A) = T$ if and only if $u(A) = v(A)$.

Digression. The concept of determination defined above is related to that of determination by a matrix, as deployed in Proposition 2.2, in the following way. If \mathbf{M} is a matrix with algebra \mathbf{A} and set of designated values D ($D \subseteq A =$ the universe of \mathbf{A}), then \vdash is determined by \mathbf{M} in the earlier sense just in case \vdash is determined by $\{v_h | h \in H\}$ where H is the set of all homomorphisms from the language of \vdash to \mathbf{A} , and for $h \in H$, the valuation v_h is defined by: $v_h(A) = T \Leftrightarrow h(A) \in D$, all formulas A .) Note that whenever \vdash is in this sense determined by a matrix, \vdash is substitution-invariant. Similarly, being determined by a class of Kripke models in the usual sense (as deployed in §3) is a matter of being determined by the class of all valuations $v_x^{\mathcal{M}}$ for models \mathcal{M} in the class, and points x in \mathcal{M} , where $v_x^{\mathcal{M}}(A) = T$ just in case $\mathcal{M} \models_x A$. **End of Digression.**

We pause to note that one possible analog for equational logic of EIP, following the analogy of ‘ $\dashv\vdash$ ’ with ‘ $=$ ’ is trivial, since what corresponds to the substitution-invariance condition is built into the definition of an equational theory. Let t_1 and t_3 be terms from the language of such a theory, which, for continuity with the discussion in the preceding section we treat as the theory of a class \mathbf{K} of algebras. Next, suppose that $\mathbf{K} \models t_1 = t_3$, and we want to know under what conditions there is an ‘interpolant’ for these *terms* (not, as in §3, formulas), meaning by this a term t_2 in which the only variables to appear are those common to t_1 and t_3 , for which we have $\mathbf{K} \models t_1 = t_2$ and $\mathbf{K} \models t_2 = t_3$. As usual, we will need also to assume that t_1 and t_3 do have at least one variable in common; but no further assumption is needed to see that the reasoning in the proof of Prop. 4.1 will always deliver a suitable t_2 . To illustrate with the simple case in which, exhibiting the variables—for which we continue with Porębska’s (somewhat unconventional) ‘ a ’, ‘ b ’, etc. from §3—used in the terms, t_1 is $t_1(a,b)$ and t_3 is $t_3(a,c)$. Then from $t_1(a,b) = t_3(a,c)$, we obtain the interpolating term t_2 by replacing ‘ b ’ in t_1 by ‘ a ’ – a special case of the (uniform) substitution of arbitrary terms in equations for variables. (If there is a partial ordering \leq associated with the algebras concerned, then we could also consider the possibility of finding suitable interpolants t_2 on the basis of inequalities $t_1 \leq t_3$, in the sense that t_2 should contain only the variables common to t_1 and t_3 and we have $t_1 \leq t_2 \leq t_3$. The argument just sketched for the case of equalities does not go through for inequalities, however, since it tacitly exploits the fact that equality is symmetric. One could also consider a version of the quasi-equational interpolation property present in connection with (12) in the preceding section with the ε_i which feature there taken as inequalities rather than equations. This, as opposed to the ‘term-interpolating’ version just considered, is what [15] calls the interpolation principle for inequalities, or IPI – see especially Theorem 1 of [15]. However, it is the ‘term’ version that is called IPI in [16], p.460.)

We shall need some preliminary observations before we can move (Thm.4.4(ii)) from EIP to IP for the equivalential fragment (of classical logic). Lemma 4.3 appears as 0.3.2(iii) in [9], which puts the point in a more general setting; cf. also, for a special case, Prop. 3.8 of [8]. (In fact [9] considers consequence relations and also generalized – or “multiple succedent” – consequence relations.) For Lemma 4.2, §0.4 and later sections of [9] provide discussion, applications, and numerous variations; see also pp.124–129 of [7] and, for a slightly different perspective, Prop. 2.6 of [8].

LEMMA 4.2 *If a consequence relation \vdash is determined by a class \mathcal{V} of valuations which is closed under the operation \sim defined above, then \vdash satisfies the following (“Exchange”) condition, for all formulas A, C , and all sets Γ of formulas (of the language of \vdash):*

If $\Gamma, A \vdash C$ then either $\Gamma \vdash C$ or else $\Gamma, C \vdash A$

Proof. For brevity, we write “ $v(\Gamma) = T$ ” to mean that for all $B \in \Gamma$, $v(B) = T$. Suppose (i) $\Gamma \not\vdash C$ and (ii) $\Gamma, C \not\vdash A$. We show that given (i) and (ii), the hypothesis in 4.2 yields the conclusion that $\Gamma, A \not\vdash B$. By (i), since \vdash is determined by \mathcal{V} , there exists $u \in \mathcal{V}$ with $u(\Gamma) = T$, $u(C) = F$; now if we don’t have $\Gamma, A \vdash B$ (which would give us our conclusion already), it must be that $u(A) = F$. Similarly, by (ii), we have $v \in \mathcal{V}$ with $v(\Gamma) = T$, $v(C) = T$, $v(A) = F$. Since \mathcal{V} is supposedly closed under \sim , $u \sim v \in \mathcal{V}$. But since $u(\Gamma) = v(\Gamma) = T$, $u \sim v(\Gamma) = T$, and as $u(A) = v(A) = F$, again $u \sim v(A) = T$. Finally, since $u(C) = F \neq T = v(C)$, we have $u \sim v(C) = F$, implying, since \vdash is determined by \mathcal{V} and $u \sim v \in \mathcal{V}$, that $\Gamma, A \not\vdash B$, as desired. \square

Though we do not need to investigate the matter here, it may be of interest to consider how (if at all) the Exchange Condition in 4.2 would need to be adjusted to yield a condition not only necessary but also sufficient for the \vdash concerned to be determined by some \mathcal{V} closed under \sim . Solutions – regrettably cumbersome in form – may be found in Theorems 3.2.1 and 3.2.2 of [9]. (In fact attention is there restricted to what [9] calls Lindenbaum consequence relations, though this is not much of a restriction in practice.)

For a simple counterexample showing that even with $\Gamma = \emptyset$, the case that matters for our use of Lemma 4.2 below, the result does not hold for the equivalential fragment of intuitionistic logic, put $A = p$, $C = (p \leftrightarrow q) \leftrightarrow q$. (As remarked in §3, the consequence statement with A and C exchanged here – the claim that $C \vdash A$ – is intuitionistically equivalent in this case to Peirce’s Law.)

LEMMA 4.3 *The class of all \leftrightarrow -boolean valuations (for any language) is closed under \sim .*

Proof. A straightforward matter of following out the consequences of the definitions of \leftrightarrow -booleanness and \sim . \square

THEOREM 4.4(i) *The consequence relation \vdash_{\leftrightarrow} satisfies the Exchange Condition of 4.2:*

If $\Gamma, A \vdash_{\leftrightarrow} C$ then either $\Gamma \vdash_{\leftrightarrow} C$ or else $\Gamma, C \vdash A$ (for all Γ, A, C).

(ii) \vdash_{\leftrightarrow} has IP.

Proof. (i): From 4.2 and 4.3, since \vdash_{\leftrightarrow} is determined by the class of all \leftrightarrow -boolean valuations. (ii): For this proof, we write ‘ \vdash ’ for ‘ \vdash_{\leftrightarrow} ’. Suppose $A \vdash C$ and that A and C share at least one variable, p . We want B for which $A \vdash B \vdash C$, the only variables appearing in B being those common to A and C . By part (i) of the Theorem, since $A \vdash C$, we have either $\vdash C$ (in the present appeal to the Exchange Condition, Γ is \emptyset), or else $C \vdash A$. In the first case, we can take the desired B as $p \leftrightarrow p$. In the second case, since here we have $A \not\vdash C$ and \vdash is substitution-invariant (*cf.* the second-last sentence of the Digression above), \vdash enjoys EIP. Thus we have a formula B whose only variables are those common to A and C , by Proposition 4.1, with $A \not\vdash B \not\vdash C$, and thus, in particular, $A \vdash B \vdash C$ as required for IP. \square

We remark that the two disjuncts in the formulation of the Exchange Condition are not mutually exclusive, as we can illustrate still in the $\Gamma = \emptyset$ case, by putting $A = p \leftrightarrow p$, $C = q \leftrightarrow q$; here, where $A \vdash_{\leftrightarrow} C$, we have both $\vdash_{\leftrightarrow} C$ and $C \vdash_{\leftrightarrow} A$. This example also serves as a reminder that we do not have the Unrestricted Interpolation Property (from §1) for \vdash_{\leftrightarrow} .

The IP for \vdash_{\leftrightarrow} seems to be rather insignificant for a reason that we can bring out using Part (i) of Theorem. 4.4, already used here to derive it (as Part (ii)). Typically the “unshared” variables – in our discussion the various q_i (in A) and r_j (in C) – which have to be discarded occur essentially in A and C (respectively). By a variable’s occurring *essentially* in a formula, relative to a given \vdash , we mean that the formula is not, according to \vdash , logically equivalent to (i.e., $\dashv\vdash$ -related to) any formula not containing that variable. For a simple but representative example, take \vdash as the consequence relation of classical or intuitionistic logic, and $p_1 \rightarrow (p_2 \wedge \neg q) \vdash p_1 \rightarrow (p_2 \vee r)$. Here the obvious interpolant is the formula $p_1 \rightarrow p_2$, and the variables (q, r) that needed to be discarded to obtain it occurred essentially in the A and C concerned. The classical equivalential case is not like this, however, as we now report; $Ess(A)$ is the set of variables occurring essentially in the formula A , relative to \vdash_{\leftrightarrow} :

PROPOSITION 4.5 *For any equivalential formulas A, C , if $\vdash_{\leftrightarrow} C$ then, if $A \vdash_{\leftrightarrow} C$:*

- (i) $Ess(A) = Ess(C)$, and further
- (ii) For any (equivalential) B such that $A \vdash_{\leftrightarrow} B \vdash_{\leftrightarrow} C$, $Ess(A) = Ess(B) = Ess(C)$.

Proof. For A and C related as hypothesized, Thm. 4.4(i) gives $C \vdash_{\leftrightarrow} A$, so A and C are logically equivalent according to \vdash_{\leftrightarrow} , the transitivity of which relation in turn guarantees that logically equivalent formulas contain exactly the same essentially occurring variables, establishing (i); (ii) follows by the reasoning in the proof of Thm.4.4(ii). \square

According to the well-known criterion of Leśniewski, for a purely equivalential formula A , we have $\vdash_{\leftrightarrow} A$ just in case each variable occurring in A occurs an even number of times. In the same vein, we can render vivid the upshot of Proposition 4.5 by noting that $Ess(A)$ consists of precisely the variables occurring an odd number of times, and that by the associativity and commutativity of \leftrightarrow (according to \vdash_{\leftrightarrow}) and the \vdash_{\leftrightarrow} -equivalence of D with $D \leftrightarrow (E \leftrightarrow E)$, any formula A can be reduced to a formula A_0 , by erasing pairs of occurrences of the same propositional variable, for which we have either: every variable in A_0 occurs exactly once, or else A_0 is of the form $p \leftrightarrow p$ for some variable p . (The latter is what we get when $\vdash_{\leftrightarrow} A$ and a final pair of occurrences cannot be erased on pain of leaving nothing behind.) If we consider all propositional variables as coming in some fixed order and further require the variables of A_0 to appear in alphabetical order, then when $A \vdash_{\leftrightarrow} C$, except in the special case that $\vdash_{\leftrightarrow} C$ outright, the reduced forms A_0 and C_0 of A and C – and B_0 of any interpolant B – will be exactly the same formula.

This may or may not seem to the reader to make IP for \vdash_{\leftrightarrow} a trivial or uninteresting matter. Our concern has not been to claim any major significance for the result itself, but to show – what seems at least to be of some interest – how the result can be obtained from a consideration of EIP for \vdash_{\leftrightarrow} along with the fact that \vdash_{\leftrightarrow} satisfies the Exchange Condition.

5. Some Questions in Closing

We end with three queries arising from the material presented above. The first is: what would a proof of IP for arbitrary fragments of classical propositional logic look like? In §1 [4] and [13] were cited as sources for (approximately) this result. Both give an \rightarrow formulation rather than a \vdash formulation, so that we are looking for an interpolant B for the implication $A \rightarrow C$ when the latter is a classical tautology. [4] is rather far from presenting the desired result because they allow the interpolant itself to be constructed with the aid of \rightarrow exploiting in particular the fact that disjunctions of formulas can be built by exploiting the classical equivalence of $D \vee E$ with $(D \rightarrow E) \rightarrow E$, so what we have is only interpolation for arbitrary fragments which include \rightarrow . The argument in [13], pp.10–11, by contrast insists that B is constructed from materials available in an arbitrarily selected fragment, which need not include \rightarrow , but the argument, which makes extensive use of (iterated) substitutions, requires the availability of constants (\top, \perp) to cope with the special case of inconsistent A or tautologous C . The latter feature could presumably be eliminated by a careful re-working of the proof, and reformulation of the result (which, as it stands, is more in the style of the ‘Unrestricted’ version of IP described in §1). Our first question, then, is as to the precise structure of the envisaged re-working.

The second question is just a rider to the first: can a ‘uniform’ version IP be established for all fragments of classical logic? Henkin showed in §5 of [5] that for full classical propositional logic (with all of some functionally complete set of connectives available, that is) we can find for any formulas A and C , the latter a classical consequence of the former, a formula B which serves as interpolant for A and C , where B depended not on C itself but only on the set of variables (our q_1, \dots, q_m) which had to be absent from B because they are absent from C . The chosen B will not only interpolate between A and C but between A and any other consequence of A from which those variables are absent. Alternatively, one can find for any given C and set of variables (our r_1, \dots, r_n) a formula B which works uniformly as an interpolant between any A not containing those variables and C . Renardel de Lavalette [24], §6.5, called these right and left interpolants, respectively, and asked whether they were forthcoming in the case of intuitionistic propositional logic, a question answered affirmatively by Pitts in Prop. 11 of [19] (though for a statement of the result which brings out its character as an answer to [24]’s question, see §2 of [10]). An affirmative answer for the case of classical logic is presumably to be recovered from the minor re-working of the argument in [13] already alluded to. (We are speaking of classical propositional logic here, of course. Henkin showed in [5] that there was no corresponding uniform version of IP for classical predicate logic, and this is what makes Pitts’ result so surprising since in many such metatheoretical respects ‘negative’ results for classical predicate logic are matched by corresponding negative results for intuitionistic propositional logic.) The proof of Proposition 4.1 above, incidentally, provides uniform interpolants in the sense of EIP. That is, given $A(p, q)$ and $C(p, r)$ for which $A(p, q) \dashv\vdash C(p, r)$, and here we take $l = m = n = 1$ for expository simplicity, we found $B(p)$ as $A(p, p)$ for which, given the substitution-invariance of \vdash , we have

$$A(p, q) \dashv\vdash B(p) \dashv\vdash C(p, r)$$

and this same $B(p)$ works for any D which, like the initially given C , contains no occurrences of q , as an interpolant between A and D . (It is easy to see we could equally well have found B on the basis of the given C , which works for any r -free A .)

Our third and final question concerns the sensitivity of matters of interpolation amongst the intermediate logics to the precise choice as to what counts as an intermediate consequence relation. Maksimova’s ([15] in particular) as well as most others cited in the course of our discussion were cast in the framework which conceives of logics certain collections of formulas perhaps answering to various closure conditions, such as closure under Uniform Substitution and *Modus Ponens* (if \rightarrow is present). The intermediate logics, for example, are thought of as sets of formulas closed under these two operations and intermediate (in inclusion) between the sets of theorems of intuitionistic and classical propositional logic. When logics are conceived as consequence relations, there is less unanimity as to what to mean by “intermediate”. The most liberal (and literal) construal would, taking for granted the intuitionistic and classical consequence relations \vdash_{IL} and \vdash_{CL} , respectively, count a consequence relation \vdash as intermediate just in case:

$$\vdash_{IL} \subseteq \vdash \subseteq \vdash_{CL}$$

For the same reasons as on the set-of-formulas conception of logics, one would normally expect to require the further condition that \vdash be substitution-invariant in order to count as an intermediate consequence relation. This is the policy of Rautenberg in [22]. On the other hand, Darsow and Kittel, in [3], adopt a quite different convention, requiring the inclusions inset above, not requiring substitution-invariance, but insisting that \vdash also satisfy further conditions corresponding to what would appear as assumption-discharging rules in a natural deduction proof system for \vdash_{IL} , including amongst others (23).

$$\text{If } \Gamma, A \vdash B \text{ then } \Gamma \vdash A \rightarrow B \quad (23)$$

Actually, in [3] the satisfaction of such conditions is built into the (accordingly non-standard) definition of ‘consequence relation’. The authors define the consequence relation \vdash_S induced by an intermediate logic S in the set-of-formulas sense by (24)

$$\Gamma \vdash_S A \text{ if and only if } \Gamma \cup S \vdash_{IL} A \quad (24)$$

and according to them the only ways (what they call) an intermediate consequence relation can fail to be \vdash_S for some suitable S are: (i) that \vdash_S is non-compact (i.e., not finitary in the sense of our §1), (ii) that \vdash_S is not substitution-invariant. More interesting for the topic of IP is the possibility, remarked on—though, it would seem, not explicitly illustrated in [22]—by Rautenberg, of an intermediate consequence relation, in the liberal sense, which does not satisfy (23). What, in detail, might a concrete realization of this possibility look like? Something to wonder about – but our official third question is, rather, this: what happens to Maksimova’s “seven intermediates with IP” result when intermediate consequence relations in this liberal sense are at issue? Does it remain intact? (We mean IP stated in terms of the consequence relation, of course, not in terms of the implicational consequences of the empty set according to that relation.)

Digression. Maksimova has considered, e.g. in [16], ‘interpolation for deducibility’ (‘IPD’) as she calls it, for normal modal logics, but – as far as I know – not addressed the question just raised *à propos* of intermediate logics. The latter property, called ‘interpolation for derivability’ in [1], Chapter 14, means for a modal logic in the ‘set of formulas’ sense, that whenever C is derivable by *Modus Ponens* and Necessitation from A together with any formulas in the logic, there is a ‘shared-variables only’ B similarly derivable from A and from which C is likewise derivable. In terms of the Kripke semantics for modal logic this corresponds to preservation of truth-throughout-a-model rather than preservation of truth-at-an-arbitrary-point-in-a-model, the

models in both cases being models for the logic. That is, by contrast with the consequence relation determined by the class of such models in the sense of the Digression in §4, the associated consequence relation here is determined by the class of all valuations $v^{\mathcal{M}}$ for \mathcal{M} in the class, where $v^{\mathcal{M}}(A) = T$ just in case for all points x in \mathcal{M} , $\mathcal{M} \models_x A$. Though diverging for modal logic, these two consequence relations, the earlier point-by-point one and the current model-by-model one, coincide for the models provided by the Kripke semantics for intuitionistic and intermediate logic. Thus the distinction between IP and IPD is not what is at issue in our question about the possibility of intermediate consequence relations not satisfying (23). The closest analogues of \vdash not satisfying (23) yet for which $\vdash_{IL} \subseteq \vdash \subseteq \vdash_{CL}$, would appear to be non-normal modal logics located \subseteq -between two normal modal logics. A semantic treatment (due to Krister Segerberg) of such logics, in terms of a simple variation on Kripke models, may be found in [1], §5.6. A corresponding treatment for the intermediate consequence relations not satisfying (23) would be most welcome. ***End of Digression.***

Though it is somewhat removed from the areas we have been considering, it is worth noting that the difference between \vdash -based and \rightarrow -based definitions of IP also rears its head for ‘contractionless’ logics such as Łukasiewicz’s many-valued logics, which typically lack IP in the sense of the \rightarrow -based definition (see [14]). Suppose \vdash is the consequence relation determined by the three-element Łukasiewicz matrix (as we have been understanding “determined by a matrix” here); (23) is not satisfied: we have $p \vdash (p \rightarrow .p \rightarrow q) \rightarrow q$ whilst $\nvdash p \rightarrow ((p \rightarrow .p \rightarrow q) \rightarrow q)$, for example. What – a sort of variant on our third question – becomes of IP for this \vdash when the \vdash -based definition of IP we have been working with is employed?

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