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Contra-Classical Logics

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This paper is dedicated to the memory of Chris Ash (1945–1995).

ABSTRACT

Only propositional logics are at issue here. Such a logic is *contra-classical* in a superficial sense if it is not a sublogic of classical logic, and in a deeper sense, if there is no way of translating its connectives, the result of which translation gives a sublogic of classical logic. After some motivating examples, we investigate the incidence of contra-classicality (in the deeper sense) in various logical frameworks. In Sections 3 and 4 we will encounter, originally as an example of what (in Section 2) we call a contra-classical modal logic, an unusual logic boasting a connective ("demi-negation") whose double application is equivalent to a single application of the negation connective. Pondering the example points the way to a general characterization of contra-classicality (Theorems 3.3 and 4.6). In an Appendix (Section 5), we look at one alternative to classical logic as the target for such translational assimilation, intuitionistic logic, calling logics which resist the assimilation, in this case, *contra-intuitionistic*. We will show that one such logic is classical logic itself, thereby strengthening a result of Wójcicki's to the effect that the consequence relation of classical logic cannot be faithfully embedded by any connective-by-connective translation into that of intuitionistic logic. (What the "faithfully" means here is that not only is the translation of anything provable in the 'source' logic provable in the 'target' logic, but that also anything whose translation is provable in the target logic is provable in the source logic.)

1. Introduction

Generally the sentential logics one encounters with connectives notated from the stock commonly used for classical logic ($\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, say, for the familiar binary connectives, $\neg$ for negation, $\top$ and $\bot$ for the nullary truth and falsity constants) differ from classical logic in refraining from endorsing certain classically accepted principles rather than in endorsing classically unacceptable principles. In the somewhat authoritarian terminology bequeathed by Quine to those who thought about such matters in the 1970s, this could be put by saying that most actually encountered *deviant* logics 'deviate' from classical logic by omission rather than by commission. (Quine [1970], Haack [1974].) We shall be calling such logics "contra-classical", because they run counter to (rather than merely falling short of) classical logic. More precisely, we call a logic *contra-classical* just in case not everything provable in the logic is provable in classical logic. Since for the moment we are thinking of the things which are or are not provable in a logic as formulas – as opposed to sequents of some more general form (see below) – this simply means that not every provable formula in a contra-classical logic is a truth-table tautology.[1]

There are, however, famous exceptions to this generalization about the rarity of deviance by commission, especially in respect of the behaviour of →. One exception is the case of connexive logic—really itself a family of exceptions since there are various ways of implementing the basic idea, which involves the provability of all formulas of the form (1*a*) or (1*b*):

(1*a*) $\neg(A \to \neg A)$ (1*b*) $(A \to B) \to \neg(A \to \neg B)$

and is therefore contra-classical since not all such formulas are tautologous. (1*a*) and (1*b*) are called Aristotle's Thesis and Boethius' Thesis, respectively, in the literature on connexive implication (see note 7). They are interderivable given rather weak ancillary assumptions about the logical behaviour of "→" and will be referred to collectively as (1) below. Another case would be the Abelian Logic of Meyer and Slaney [1989], whose implicational fragment extends the familiar subclassical system of *BCI* logic[2] by the highly unfamiliar 'Abelian' axiom(-schema) given here as (2):

(2) $((A \to B) \to B) \to A$

As with (1), (2) is not generally tautologous, so Abelian logic gives us another example of a contra-classical logic. Of course while (2) itself is unfamiliar, putting ⊥ for 'B' gives us something to which we are much more accustomed. It is this consideration that provides Meyer and Slaney's motivation in developing Abelian Logic: to obtain various classical properties for negation when the negation of A is defined as $A \to \bot$ and nothing special is assumed about ⊥ – nothing that does not hold for every formula, that is. If we want, in particular, a double negation elimination principle to be forthcoming, then we must have (2) in full generality. (The authors are well aware of what we are calling the contra-classicality of their logic, putting the point by saying that the logic is not "cool for Boole".) In view of the Post-completeness of classical propositional logic – to which we return at the end of this section – adding classically invalid principles while remaining within the language of classical propositional logic will, on pain of inconsistency, mean giving up some classically valid principles. So, in the above terminology, going counter to classical logic in some respects will typically necessitate also falling short of classical logic in others.

In describing these examples as contra-classical we have taken the formulas involved 'at face value': for the claim that such-and-such a theorem of a non-classical logic is not classically provable (or tautologous) we take the identity of the connectives involved as given by the way they are notated. If someone were to propose a logic in what is notationally at least the same language as that of classical logic (for definiteness, taken with the seven connectives listed above as primitive) but in which ∨, as well as ∧, had exactly the properties enjoyed by ∧ in classical logic, so that for

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[1] As this sentence illustrates, we tend to speak of arbitrary logics in syntactic terms, referring to elements of a logic as provable in (or 'theorems of') the logic, though in the case of classical (propositional) logic we exploit the equivalence of provability in any standard proof-system for the logic with its semantic correlate of tautologousness (truth-table validity). What we have just defined as contra-classicality in the text is called non-classicality "in the full blooded sense" at p.197 of McCall and Vander Nat (1969); see further note 11 below.

[2] The '*B*', '*C*', and '*I*' are labels derived from combinatory logic for the schemata:
$(A \to B) \to ((C \to A) \to (C \to B))$,
$(A \to (B \to C)) \to (B \to (A \to C))$,
$A \to A$,
respectively. *BCI* logic is named after its best known axiomatization, in which all instances of these schemata are taken as axioms, and the sole rule of proof is the rule *Modus Ponens* (licensing the transition from $A \to B$ and A to B).

all formulas A and B,[3] (3) as well as its converse, and (4), though not its converse, for example, were provovable,

(3) $(A \vee B) \rightarrow (A \wedge B)$

(4) $(A \vee B) \rightarrow A$

we should probably not think much of this as a proposed contra-classical logic, at least in the absence of some special informal argumentation as to why we should think of $\vee$ as representing any kind of disjunction. Such special background is indeed provided in the connexive implication and Abelian Logic cases mentioned earlier, but rather than rest a definition of contra-classicality on such vague extra-systematic considerations, we should regard the above definition as picking out a particularly superficial version of that notion.[4] The contrast would be with a less superficial notion of a contra-classical logic, still conceived of – for the moment – as a certain set of formulas (the provable formulas of the logic), S, say, defined as follows: S is *profoundly contra-classical* if there is no translation of the language of S into the language of classical propositional logic in such a way that the translations of all theorems of S are classical tautologies. Here, by a *translation* is meant a mapping τ which is the identity map on propositional variables and such that for each *n*-ary connective # of the language of S, there is an *n*-ary (not necessarily primitive) connective of the language of classical logic $\#^\tau$ satisfying for all formulas $A_1,\ldots,A_n$:

$$\tau(\#(A_1, \ldots, A_n)) = \#^\tau(\tau(A_1),\ldots,\tau(A_n)).$$ [5]

According to this definition, the envisaged $\vee$-treated-like-$\wedge$ logic fails to count as profoundly contra-classical since setting $\#^\tau = \#$ for all (primitive) # except for $\vee$, and $\vee^\tau = \wedge$, determines a τ which maps every theorem of the logic to a classical tautology. In this case, the function τ is surjective, but that was not required by the definition. Still more importantly not part of the definition is the converse direction of the implication we do require; denoting the set of tautologies by CL, S is contra-classical just in case there is no translation τ as above for which (5) holds for all formulas A of the language of S:

(5) $A \in \mathsf{S} \Rightarrow \tau(A) \in \mathsf{CL}$

And (5) is a much weaker condition on translations considered for purposes of translationally

[3] The letters 'A', 'B', etc. are schematic for arbitrary formulas of whatever language is under consideration (and , below, 'Γ', 'Δ', for sets of such formulas) All language will be presumed to be based on the same countable sequence of propositional variables (or "sentence letters") $p_1, p_2, \ldots$ and to differ only as to what connectives are available for constructing formulas from this common basis. For convenience, we will write the first four variables from this list as *p*, *q*, *r*, *s*.

[4] Thus Angell [1962] develops an early connexive logic remarking (p.327) that "The expression 'if…then…' would be assigned as the interpretation of a primitive symbol '→'", but this is clearly a point about motivation and not part of the formal development (or indeed of any precise semantical account of the formalism).

[5] Since there are a number of policies that may be taken towards the question of defined (or definable) connectives, some explanation is in order as to what we have in mind with the parenthetical "not necessarily primitive" in this sentence. Where $A(p_1,\ldots,p_n)$ is a formula in which only the propositional variables exhibited occur, then we take ourselves to have available a defined *n*-ary connective # where $\#(B_1,\ldots,B_n)$ is the resulting of substituting B_i for p_i ($i = 1,\ldots,n$) in $A(p_1,\ldots,p_n)$. On this policy, a new connective is not actually added to the object language, "$\#(B_1,\ldots,B_n)$" simply being a convenient metalinguistic device for referring to the formula resulting from the substitution just indicated. Alternatively, one may conceive actually adding a new connective to the object language, in which case the description of what is going on with the translations has to be complicated by considering them as translations into a 'definitional extension' of what we are treating them as translationally embedding into. See the passages of Wójcicki [1988] cited in Section 5 below for an example of the latter procedure (though in addition Wójcicki is concerned specifically with what we shall refer to presently as faithful embeddings and we are not); a more general discussion of the different ways definitions of connectives have been considered in the literature, together with references thereto, may be found in Section 3 of Humberstone [1998].

embedding one logic (the "source" – here, the unspecified S) in another (the "target" – here CL), for which what is usually imposed is the two-way version (6), sometimes described in terms of the embedding being's *faithful* (or 'exact'):

(6) $\quad A \in \mathsf{S} \Leftrightarrow \tau(A) \in \mathsf{CL}$

For example, Prawitz and Malmnäs [1968] would say that (6) – and not merely (5) – renders S 'schematically interpretable' in CL. (The 'schematically' part refers to the way the translation τ operates on compound formulas, as above, via $\#^\tau$. These authors do not impose the requirement that τ is the identity map on propositional variables. And why should they? They are surveying a well established logical industry interested in extracting information about one logic by finding it duplicated in however ingenious a disguise within another logic. The translation describes the disguise; for instance the p_i in the source appear disguised as $\neg\neg\, p_i$ or as $p_i \vee \bot$ or as $\Box p_i$ in the target, to recall three well-known examples – all discussed by Prawitz and Malmnäs. But here we are interested in reinterpreting the primitive connectives of one logic in terms of those of another. Our translations accordingly leave the propositional variables as they stand. Related precisifications of the informal idea of a translational embedding may be found in Tokarz and Wójcicki [1971] and Wójcicki [1988], in which schematic translations leaving the propositional variables undisturbed are called *definitional* translations, and to which we shall return in Section 5. There is also a general discussion of the matter in Chapter 10 of Epstein [1990].[6]) The contrast between (5) and (6) can be put by reading (*via* the action of τ) the usual truth-table semantics of the language of CL into the language of S, (6) calls for soundness and completeness of S with respect to that semantics, while (5) asks only for soundness – that every $\tau(A)$ should be tautologous. If instead we defined a logic to be contra-classical when there was no translation τ satisfying (6), this would actually make no difference to the situation – described in Proposition 1.1 below – in respect of profound contra-classicality amongst logics conceived of as sets of formulas, but an analogous move in other logical frameworks (introduced below) would make a great difference, as it would also when applied to what we shall later call contra-classicality *modulo* this or that set of connectives. A special case concerns the contra-classical modal logics discussed in the following section (some terminology from which is employed here). Since for every of the 1-place truth-functions f we have $f = f^3$, any candidate for the role of $\Box^\tau$ will be such that $\Box^\tau\Box^\tau\Box^\tau p \leftrightarrow \Box^\tau p \in \mathsf{CL}$, so (for instance) the smallest normal modal logic would count as contra-classical on the rival understanding of this phrase (with (6) in force), since $\Box\Box\Box p$ and $\Box p$ are not provably equivalent in that logic. Yet some of its extensions, in particular the two Post-complete normal modal logics, on the rival conception, would not count as contra-classical. Since we thus lose contra-classicality on passing from weaker to stronger logics, it is clear that the rival conception does not embody the idea with which we began of contra-classicality as deviation by commission (rather than mere omission). But here we are digressing from the current theme of 'profound' contra-classicality.

Clearly, to return to our original examples, the mere provability in a non-classical logic of all

[6] A section of this chapter – apparently co-authored by Epstein and S. Krajewski – has the title "Logics which cannot be translated grammatically into classical logic". The authors are not discussing contra-classical logics, however, because what they understand by a translation is the biconditional version ("faithful" translational embedding) rather than the one-way version of the notion (along the lines of (6) rather than (5), that is – though they are working in what we shall shortly call the framework SET-FMLA rather than FMLA). It is worth pointing out that certain transformations which do not count as translations even when both the "schematicity" and "leave the propositional variables alone" are relaxed, may also be used to throw light on one logic by relating it to another, such as the transformation which simply prefixes a double negation once and for all to the outside of each whole formula (rather than on its proper subformulas as it is constructed), relating intuitionistic to classical logic *via* Glivenko's Theorem.

formulas of the form (1*a*,*b*) above, or all formulas of the form (2), does not suffice for the profound contra-classicality of that logic, since we can in both cases choose $\rightarrow^{\tau}$ as $\leftrightarrow$ (and, say, $\neg^{\tau} = \neg$ for the case of (1)).[7] Indeed, still thinking of a logic as a set of formulas but adding that this set should at least be closed under the uniform substitution of arbitrary formulas for propositional variables, it is easy to see that there are no consistent profoundly contra-classical logics at all.[8] By the consistency of S here we mean simply that not every formula of the language of S is provable in S. If S is not consistent in this sense, then *p*, for example (see note 3) is provable in S and since for every translation τ, $\tau(p) = p$, τ maps a theorem of S to a non-theorem of classical logic. The remainder of the claim just made we state – for ease of back-reference later – as Proposition 1.1. For its proof we need some defined connectives of the language of classical logic (taking as primitive – say – those listed in our opening sentence). We define, for each *n*, the connective $\top_n$ of the language of classical logic thus:

$$\top_n(A_1,\ldots,A_n) \;=\; A_1 \rightarrow (A_2 \rightarrow \ldots \rightarrow (A_n \rightarrow \top)\ldots)$$

understanding the right-hand side to be $\top$ itself when $n = 0$. (Any other way of defining the linguistic embodiment of the 'constant true' truth-function of *n* arguments would do equally well.) The parenthetical words in the statement of Proposition 1.1 will be clarified below.

PROPOSITION 1.1 *A logic* S *(in FMLA) is profoundly contra-classical if and only if* S *is inconsistent.*

Proof. The 'if' direction we have already seen. For the 'only if' direction, note that if S is consistent then since none of the p_i can then be S-provable, any theorem of S is of the form $\#(A_1, \ldots, A_n)$ for some connective # of the language of S. To show that S is not contra-classical, we need a translation, τ, which takes every composite formula of the language of S such formula to a truth-functional tautology. The desired τ is given by: $\#^{\tau} = \top_n$ for each *n*-ary connective # of the language of S, with $\top_n$ as defined above. ◗

Thus if we are interested in *consistent* profoundly contra-classical logics, we had better operate with something other than a set of formulas as our conception of what a logic is. As is well known, there are numerous alternatives. One way of summarizing the most significant options is to

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[7] For the whole of the implicational fragment of Abelian logic, which may be axiomatized by adding (2) as an axiom schema to the basis for *BCI* logic given in note 2, this same translation works. (Note that in any instance of *B*, *C*, or *I*, each propositional variable occurs an even number of times, so rewriting the '→'s as '↔'s yields tautologies, and that *Modus Ponens* with '↔' replacing '→' preserves this property. However, Meyer and Slaney [1989] place demands on conjunction and disjunction, for example, which certainly do not go across into tautologies when these connectives are translated by themselves and '→' by '↔'. In the connexive logic case, the same is true for the axioms given on p.328 of Angell [1962]. responding to some difficulties for the connexive logics of Angell's and his own in the 1960s, McCall ([1975], p.442) presents a new system, all of whose axioms yield tautologies when '→' is replaced by '↔' and the boolean connectives are left as they stand. Under a somewhat different replacement, Meyer [1977] shows that McCall's system can be embedded faithfully (and not just in the weaker way we are considering) into a range of normal modal logics (he selects **S5** as a representative target); similar investigations, with a somewhat different – though still recognisably "connexive" – source logic and a very different translation, are undertaken in Pizzi and Williamson [1997]. There seems to have been a disproportionately high Australasian interest in connexive logic – witness also Routley and Montgomery [1968], Routley, Plumwood *et al.* [1982] – in particular §4 of Chapter 2 – and Mortensen [1984]. In fact the practice of calling (1*a*) Aristotle's Thesis appears to stem from this tradition. McCall's original use of the phrase was for the (obviously closely) related principle: $\neg(\neg A \rightarrow A)$.

[8] The status of closure under uniform substitution as a condition on logics has frequently been contested (the connexive logic proposed in McCall [1975] does not satisfy it, for example), but we prefer not to enter into a discussion of the matter here.

introduce the notion of a sequent for a language, coming itself in several versions – a choice amongst we call a choice of *logical framework* – thus: if Γ, Δ, are finite sets of formulas of the language under consideration, then a pair ⟨Γ, Δ⟩ which we may write in some more suggestive notation, such as "Γ ≻ Δ" is a *sequent* over that language

- for the logical framework SET-SET (no conditions on Γ, Δ)
- for the logical framework SET-FMLA if Δ = {B} for some formula B
- for the logical framework FMLA if Δ = {B} for some formula B and Γ = ∅

Obviously there are many further variations possible and indeed alive and well in the logical literature. Instead of having finite sets we could take arbitrary – and not just finite – sets, or instead sequences, or multisets of formulas, for example, or we could insist that Γ is non-empty, or that each of Γ, Δ, consists of a single formula (not just Δ, as for SET-FMLA), and so on. The basic idea is that choosing a logical framework is choosing a type of syntactic entity to be provable or unprovable in the logics one wishes to investigate. The framework FMLA amounts to taking these entities to be simply the formulas themselves (on identifying the sequent ∅ ≻ A with the formula A), and it has especially been associated with the Hilbert-style axiomatic approach to logic, which SET-FMLA and SET-SET are descendants of Gentzen's approaches – natural deduction (for SET-FMLA in particular) and the sequent-calculus (both SET-FMLA and SET-SET). For SET-SET and SET-FMLA, certain structural rules (rules not pertaining to any particular connectives, that is) will often be insisted on, as well as the requirement of closure under uniform substitution (of arbitrary formulas for propositional variables within any provable sequent), already mentioned for the framework FMLA in which we have been conducting the discussion of contra-classicality in this section. The structural rules alluded to typically correspond in an obvious way to the various conditions used to define the notion of a consequence relation (in the case of SET-FMLA) or generalized ("multiple-conclusion") consequence relation (in the case of SET-SET), so much of the effect of refining one's conception of a logic in stepping out of the framework FMLA and into SET-FMLA or SET-SET can be obtained by deciding to identify logics with consequence relations or generalized consequence relations, respectively. We will consider such possibilities in Section 4, for the moment remaining firmly within FMLA.

Even sticking to the framework FMLA, in which we take a logic as a set of formulas, we need not take any and every such set of formulas to constitute a logic. We have already insinuated a further condition of closure under uniform substitution, but for many purposes further conditions are often added, pertaining to specific connectives. One effect of these is to partially recapture the resources of working in some richer framework (in particular SET-FMLA), and the main such condition, which presumes that '→' is one of the connectives of the language under consideration, is the further condition of being closed under *Modus Ponens*. (This is built into the usual definition, for example, of intermediate logics and—as we shall recall in the following section—modal logics.) Let us call sets of formulas which are closed not only under uniform substitution but also *Modus Ponens*, *mp-logics*, continuing to use 'logic' unmodified to allude to the requirement only of closure under uniform substitution. The logics in a given language form a lattice (with inclusion as the associated partial ordering) whose top element is the inconsistent logic (in FMLA, the set of all formulas of the language in question). The dual atoms of this lattice are the Post-complete logics. It is well known that if we consider the lattice of all *mp*-logics in the language with connectives mentioned in our opening sentence, classical logic (the set of all tautologous formulas in this language) is Post-complete, and slightly less well known that if we consider instead the lattice of all logics, classical logic is not Post-complete. The latter point was well made in Hiż [1959], with an axiomatically presented logic all of whose rules of proof were substitution-invariant, in the sense explained in Section 3 below. Since the provable formulas are precisely the tautologies, this is an *mp*-logic.[9] But further non-tautologous axioms may be added without yielding inconsistency, giving rise to consistent proper extensions which are not *mp*-logics. It would accordingly be clearer if one said that classical logic was Post-complete *qua mp*-logic rather than simply Post-complete *tout court*.[10] For this point about Post-completeness, it would suffice to consider, rather than *mp*-logics, logics closed under the passage from a formula and its negation to an arbitrary formula – *efq*-logics, say (since this is the rule sometimes called *Ex Falso Quodlibet*): classical logic would then emerge as a Post-complete *efq*-logic. But *Modus Ponens* also has a rather different status from its role as a rule of proof (whose admissibility is a closure condition on logics, that is), namely, its role as a rule of *inference*, and it is here that the connection with logic in SET-FMLA appears. (A Uniform Substitution rule, by contrast, plays only the former role. The terminology of rules of proof *vs.* rules of inference was introduced in Smiley [1963]. It has unfortunately not caught on widely.) Then the rules of inference in FMLA can be formulated as sequent schemata (zero-premiss sequent-to-sequent rules) in SET-FMLA, with premisses on the left and conclusion to the right of the separator ≻. The interesting question of how best to draw a rule of inference/rule of proof distinction within SET-FMLA or SET-SET will not detain us here.

Our original ('superficial') definition of contra-classicality and the 'profound' variation occupy

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[9] Recall that an *mp*-logic requires closure of the set of theorems under *Modus Ponens*. In terms of an axiomatization of the logic, this means only that *Modus Ponens* is an admissible rule, not that it is a derivable rule – and it is clear from the present summary of Hiż [1959]'s axiomatization that *Modus Ponens* is not derivable. A simpler example making the same point (though without the explicit attention to Post-completeness) may be found in Porte [1960]. We should also remark, for accuracy, that Hiż works in a language whose primitive connectives are only implication and negation, and not with the full panoply of connectives listed in our opening sentence.

[10] A similar point is made on p.436 of Chagrov and Zakharyaschev [1997]: "Of course, Post completeness of *L* depends essentially on the chosen lattice of logics". The point is easily overlooked in the case of modal logic, where the lattices most commonly chosen for investigation do not render this lattice-relativity crucial. The example considered above gives two such lattices, of Φ-logics (*mp*-logics in our example) and of Ψ-logics (arbitrary logics) with all Φ-logics being Ψ-logics, and a particular Φ-logic (CL) which is Post-complete *qua* Φ-logic but not Post-complete *qua* Ψ-logic. If we consider the modal case, with Φ-logics as normal modal logics and Ψ-logics as arbitrary modal logics (as these terms are defined in the following section), then there are no corresponding cases of the above pattern: there are no consistent modal logics to be found properly extending the logics which are Post-complete *qua* normal modal logics. It would be useful to have a general account of when the pattern can arise and when it cannot. (It should be pointed about that proof-systems – which in FMLA amounts to: axiomatizations – can be described as Post-complete without any such further relativity, meaning by this that they have no consistent proper extensions closed under the primitive rules of the proof-system. This would mean that there can be two proof-systems with the same theorems but which differ as to Post-completeness. Such is the policy of the papers cited in the preceding note, as well as – for example – McCall and Vander Nat (1969), in which the authors talk explicitly of Post-complete *axiomatizations*. Alternatively, one may de-relativize while still ascribing Post-completeness to the sets of provable formulas rather than to a specific proof-system, by letting the class of all admissible rules play the role just envisaged for the class of primitive rules. This gives Chagrov's concept of 'general Post completeness', on which there are several results in Chapter 13 of Chagrov and Zakharyaschev [1997].) The sensitivity to rules is also reduced in SET-FMLA and especially SET-SET, where one can have closure under the basic structural rules in the background without an excessive parochiality to particular languages and their stocks of connectives. Classical logic with all the connectives of our opening paragraph is Post-complete in both these frameworks. (The point is somewhat sensitive to the available connectives. For example, if we considered just the $\{\wedge,\vee\}$-fragment of classical logic in either of these frameworks, it has what we shall in Section 4 call constant-valued though consistent proper extensions and so is not Post-complete.) *Note*: throughout when we say "modal" we man "monomodal". The situation in respect of Post-completeness for logics in which there is more than one primitive □-operator is quite different. See for example the discussion of the bimodal case in Williamson [1998*b*], in which normal bimodal logics which in the above terminology would be called Post-complete *qua* modal logic and Post-complete *qua* normal modal logic are called respectively Post-complete and maximal consistent normal logics.

opposite ends of a certain spectrum. Where $K_{bool} = \{\wedge, \vee, \rightarrow, \leftrightarrow, \neg, \top, \bot\}$, we say that a translation τ, understood as above, is *K-fixed*, for $K \subseteq K_{bool}$, just in case $\#^{\tau} = \#$ for each $\# \in K$. This means that there is room for 'reinterpretation' only of the non-boolean connectives – the boolean connectives, taken here as those in the set K_{bool} are not to be tampered with. A logic S is, we may say, *contra-classical modulo* $K \subseteq K_{bool}$ when there is no K-fixed translation mapping all of its theorems to tautologies. The notion of contra-classicality with which we began was contra-classicality *modulo* K_{bool}; the notion of profound contra-classicality with which we replaced it amounts to contra-classicality *modulo* $\varnothing$. We return to the latter phenomenon in Section 4. The early examples—featuring (1) and (2)—of the former phenomenon concerned logics in languages which were sublanguages of the language of classical logic: their connectives were all drawn from K_{bool}, despite their rather non-boolean behaviour in the logics concerned. An interesting further possibility remains to be explored therefore, namely, the possibility of a contra-classical logic *modulo* K_{bool}, even though all connectives in K_{bool} are living up to boolean expectations (unlike, for instance, $\rightarrow$ in connexive logic). This means there will have to be *additional* connectives present, which are resistant to translation into the language of classical logic – resistant to a truth-functional interpretation, we might say. (We will be more precise about what truth-functionality involves in Section 4.)

What we have called contra-classicality, profound or otherwise, may not seem the only property deserving that name. One might expect a contra-classical logic to be required to have amongst its theorems (we are thinking of FMLA here) the negation of some classical tautology – a stronger requirement than our initial requirement that amongst the theorems there should be some formula which is not a classical tautology. But, even setting aside the question of arriving at a logic-neutral notion of negation with which to wield such a criterion, the idea seems at odds with the "universal" interpretation of propositional letters imposed by the requirement of closure under uniform substitution and distinguishing logics in FMLA from arbitrary theories in a propositional language. (If we were employing explicit propositional quantifiers, matters would perhaps be otherwise. One could disagree with a the claim that $\forall p(p \vee \neg p)$, for example, by asserting its negation, whereas what one does by asserting the negation of the quantifierless version, $p \vee \neg p$, is to commit oneself rather to $\forall p \neg(p \vee \neg p)$. Another idea that might repay investigation is the idea of a logic which is incompatible with classical logic in the sense of not being consistently combinable with classical logic. For example, identifying a SET-FMLA logic with the associated consequence relation (see Section 4) and, for consequence relations $\vdash_1$ and $\vdash_2$ denoting by $\vdash_1 \uplus \vdash_2$ the least substitution-invariant consequence relation on the language with, for its connectives, all those of the language of $\vdash_1$ and of $\vdash_2$, we could call $\vdash$contra-classical in this "combinability" sense when $\vdash_{CL} \uplus \vdash$ is inconsistent. (Here $\vdash_{CL}$ is the relation of tautological consequence, and the substitution-invariance of a consequence relation $\vdash$ means that whenever $\Gamma \vdash A$, then $s(\Gamma) \vdash s(A)$ where $s(\Gamma)$ is the result of applying some substitution to all elements of Γ and $s(A)$ is the result of applying it to A. (No confusion should result from applying this same terminology of substitution-invariance to consequence relations and also – as foreshadowed above – to rules.) This would be the superficial version of the idea. To get the profound version, first replace $\vdash$ by a consequence relation, call it $\vdash'$, in a language whose set of connectives is disjoint from that of $\vdash_{CL}$, and then demand, for the contra-classicality of $\vdash$, that $\vdash_{CL} \uplus \vdash'$ be inconsistent. There are certainly instances of consistent but contra-classical consequence relations in this sense, such as the constant-valued logics of Section 4 below, but the author has no interesting examples or results, and for this reason we do not pursue the combinability-based version of contra-classicality further here. However, we will consider briefly, in an Appendix (Section 5) to the main body of this paper, what happens when another logic takes the place of classical logic in the preferred definition of contra-classicality.

2. Contra-Classical Modal Logics

The simplest case of a language in which to investigate the possibility just raised would be that of a language in which there is a single additional connective present other than those in K_{bool}, and that connective is singulary, and the logic in question is an extension of classical propositional logic. That means that we are dealing with a modal logic in the broad sense of the term (as in Makinson [1971], for example). We take the language of such a logic to be based on the connectives in K_{bool} together with the additional 1-ary connective $\Box$. A modal logic is – recall that we continue to work in FMLA – a set of formulas in this language which includes all tautologies and is closed under *Modus Ponens* and uniform substitution. (Since the main motivating examples in Section 1 of contra-classicality involved a binary connective ($\to$) and here we are making life easier for ourselves by considering the singulary analogue, with '$\Box$' as a candidate one-place connective. Obviously a fully general discussion should proceed by allowing '$\Box$' to be *n*-ary, for some natural number *n*, and considering a modal logic in the language with such a '$\Box$' to be, as above, a set of formulas containing all tautologies and closed under *Modus Ponens* and uniform substitution. Following this more general path, however, complicates matters when it comes to various formulations, and we shall indeed consider the general case when we at the same time drop the background assumptions about the boolean connectives in Section 4.) Some investigations bearing on the question of contra-classicality amongst the modal logics were reported in Makinson [1971], and much of the following terminology is taken, sometimes with minor adaptations (as in Segerberg [1982]), from there. A modal logic S is *congruential* if whenever $A \leftrightarrow B \in \mathsf{S}$, we have $\Box A \leftrightarrow \Box B \in \mathsf{S}$, and *normal* if (7) always implies (8) for all formulas $A_1,\ldots,A_n$, and B:

(7) $(A_1 \wedge \ldots \wedge A_n) \to B \in \mathsf{S}$ (8) $(\Box A_1 \wedge \ldots \wedge \Box A_n) \to \Box B \in \mathsf{S}$

where in the case of $n = 1$, the formulas mentioned in (7) and (8) are just the consequents of the schematically represented conditionals. Further S is *monotone* when the implication from (7) to (8) always holds when $n = 1$, and *antitone* when $A \to B$'s being in S implies that $\Box B \to \Box A$ is in S. These last two notions are defined here only for making intelligible some parts of a result of Makinson's (Theorem 2.2 below), and not for their direct bearing on contra-classicality, for which the important concepts in our discussion are congruentiality and normality. The non-normal modal logics are just those which are not normal. By a *contra-classical modal logic* we will understand a modal logic which is contra-classical *modulo* K_{bool}. This means that there is no way of giving a classical interpretation to '$\Box$' will which interpret all the theorems (i.e., all $A \in \mathsf{S}$) as tautologies. Obviously, the inconsistent modal logic is contra-classical in this sense, so the question of most immediate interest becomes: are there any consistent contra-classical modal logics?

To return an affirmative answer to this question, we recall the little-discussed non-normal modal logic S6, on which – for information going beyond that provided here – see Hughes and Cresswell [1968], pp.267ff, 281–4.[11] An axiomatic description is as follows: take the closure under *Modus*

[11] *Warning*: Hughes and Cresswell [1996] is no substitute, containing only a single brief reference to S6. A slightly different semantic description of S6 may be found on p.99 of Chagrov and Zakharyaschev [1997]. The crucial role of axiom scheme (A5) in what follows is played by the same principle in a different way in McCall and Vander Nat [1969]. They consider non-normal modal logics in which strict implication is taken as primitive, and, representing the latter here by "$\to$", $\Box A$ is defined as $\neg A \to A$. When (A5) is rewritten in accordance with this definition, it has – now reading the "$\to$" as material implication – non-tautologous instances, attesting at once to contra-classicality in our original 'non-profound' sense. (*Cf.* note 1 above.)

Ponens and the rule ("Becker's Rule") licensing the transition from any formula of the form $\Box(A \to B)$, to the corresponding formula $\Box(\Box A \to \Box B)$, of the set of formulas instantiating the following axiom schemes:

(A0) $\Box A$ for any substitution-instance A of a tautology (A1) $\Box A \to A$

(A3) $\Box(\Box A \to A)$ (A4) $\Box(\Box(A \to B) \to (\Box A \to \Box B))$ (A5) $\neg\Box\Box A$

Hughes and Cresswell describe a version of the Kripke semantics for non-normal modal logics which we will not repeat here, which supplies a class of models with "normal" and "non-normal" worlds as well as a reflexive accessibility relation, and a notion of validity which amounts to truth at each of the normal worlds in any such model. The most immediately significant facts for our purposes are that not every formula of the language is valid, and that the valid formulas are exactly the theorems of S6 as just axiomatized. In particular, then, S6 is consistent. But further, we have:

PROPOSITION 2.1 *S6 is a consistent contra-classical modal logic.*

Proof. $\Box\top \wedge \neg\Box\Box\top$ is provable in S6 (by (A0) and (A5)). There are (to within classical equivalence) four candidates to be $\Box^{\tau}$: (i) $\Box^{\tau}A = A$, (ii) $\Box^{\tau}A = \neg A$, (iii) $\Box^{\tau}A = \top_1(A)$, (iv) $\Box^{\tau}A = \neg\top_1(A)$. With (i) and (iii) the first conjunct but not the second, of our S6 theorem, is tautologous, whereas with (ii) and (iv), the second conjunct but not the first gets translated into a tautology. So there is no translation (fixed on the boolean connectives) which turns the whole conjunction into a tautology. ◗

(A reader leafing through these pages and stopping at the above point may wonder what is going on: S6 is a consistent extension of classical non-modal propositional logic, after all, so by what possible justification can it be described as "contra-classical"? Well, in Section 1 we were led to a certain 'uninterpretability' based notion of contra-classicality by consideration of certain examples, and it is under this notion that S6 falls. We are not dealing with a notion of contra-classicality – briefly sketched at the end of Section 1 – according to which this amounts to the inability to be, in a sense explained there, consistently combined with classical logic.)

Having listed, in the proof of Proposition 2.1, the four boolean candidates for the role of translating '$\Box$', we can make the following further observation, which depends only on the fact that there are finitely many candidates (and so would be operative in the more general setting mooted in the preceding paragraph). If S is a contra-classical modal logic, then it is always possible to 'localize' the contra-classicality in the sense of finding some formula in which there are occurrences of '$\Box$' which resist a truth-functional construal. For there must be a formula A_1 for which $\tau(A_1)$ is not a tautology with τ as in (i) of the above proof, A_2 with $\tau(A_2)$ non-tautologous taking τ as in (ii), and similarly formulas A_3 and A_4 corresponding to (iii) and (iv). But in that case we have, in the formula $A_1 \wedge A_2 \wedge A_3 \wedge A_4$ a 'resistant' formula, provable in S but defying assimilation in CL. Of course, it may be that some subconjunction will already do the trick (as the proof of Prop. 2.1 illustrates for case of S6 itself, indeed).

Since it is well known that that every consistent normal modal logic is a sublogic of either the least such logic containing $A \leftrightarrow \Box A$ for every formula A or else that containing $\Box A$ itself for every A (these being respectively the identity logic and the unit logic in the terminology from Makinson we shall be employing presently, also called the 'Verum' and 'Trivial' systems in the literature), each of which forces a truth-functional interpretation onto '$\Box$', there is no contra-classicality to be found amongst the normal modal logics. Whilst normality implies congruentiality, the converse is not the case, raising the possibility that it is not so much the non-normality of S6 that permits it to be contra-classical, as its non-congruentiality. For example, we have $\top \leftrightarrow \Box\top$ but not $\Box\top \leftrightarrow \Box\Box\top$ provable in S6, a violation of the congruentiality condition clearly pertinent to the proof of Proposition 2.1. (In terms of the semantics alluded to, the reason it can happen that A $\leftrightarrow$ B is valid while $\Box$A $\leftrightarrow$ $\Box$B is not, is that validity means truth at all the 'normal' worlds in every model, and agreement in truth value between A and B at all such worlds does not suffice for a similar agreement in respect of $\Box$A and $\Box$B, since the truth-value of a $\Box$-formula at a normal world depends on the truth-value of the formula in the scope of the '$\Box$' at arbitrary – *including non-normal* – accessible worlds.)

Perhaps, then, congruentiality itself already suffices to rule out consistent contra-classicality. To investigate this possibility, we come now to the Makinson-style result promised. (Building on the discussion which follows, we shall be in a position to revise this tentative hypothesis about congruentiality, when we encounter, in the next section a concrete example of congruential consistent contra-classicality.) First, by way of introduction, we report without proof, on the several embedding results of Makinson [1971]. Makinson uses the following terminology for the four modal logics we get by interpreting $\Box$ as a singulary truth-function (corresponding respectively to the translations described as (i)–(iv) in the proof of Prop. 2.1): the identity logic, the complement logic, the unit logic, the zero logic. Makinson's embedding result then runs as follows.

THEOREM 2.2 *Let* S *be a consistent modal logic. Then* (*i*) *if* S *is congruential and* $\Box\top \wedge \neg\Box\bot \in$ S, S *is a sublogic of the identity logic,* (*ii*) *if* S *is monotone then* S *is a sublogic of the identity logic, or the zero logic, or the unit logic,* (*iii*) *if* S *is antitone then* S *is a sublogic of the complement logic, or the unit logic, or the zero logic.*

We summarize Makinson's proof of part (i) of the above result, since it provides a way of giving analogous results for inclusion in the other three logics. (The result appears as Theorem 1, in Makinson [1970], p.253.) The hypothesis that S is congruential means we can form the Lindenbaum algebra of S, with elements as the equivalence-classes [A] (= {B| A $\leftrightarrow$ B $\in$ S} of formulas A of the language of S, which algebra can—because of the definition of 'modal logic'—be taken to be a boolean algebra with an additional operation corresponding to '$\Box$' in the way that the join, meet, and complement correspond to '$\vee$', '$\wedge$', '$\neg$' (taking the join of [A], [B], to be [A $\vee$ B], etc.), and with unit (1) and zero (0) being the equivalence classes of the theorems of S and of their negations, respectively. We notice that since $\Box\top \wedge \neg\Box\bot$ is amongst these theorems the operation associated with '$\Box$' maps 1 to 1 and 0 to 0, and since {1,0} is obviously closed under the boolean operations, the subalgebra generated by {1,0} has just {1,0} as its universe and we get an extension of the original logic S by considering the formulas valid in (i.e., whose corresponding algebraic terms always evaluate to the element 1) this subalgebra, in which the $\Box$-operation is the identity function. That is, S is a sublogic of the identity logic. (A purely syntactical proof of this result may be found in Williamson [1998*a*], p.106, where the case of logics extendable to the unit logic is also argued syntactically. See further Williamson [1998*b*].)

By similar reasoning – either as in our sketch of Makinson's proof, or in the above 'dealgebraized' version just alluded to, we can establish similar results for the complement, unit, and zero logic, and we combine all four (with repetition of the case of the identity logic, for the sake of complete coverage) thus:

THEOREM 2.3 *Let* S *be a consistent congruential modal logic. Then*

(*i*) *if* $\Box\top \wedge \neg\Box\bot \in$ S, S *is a sublogic of the identity logic,*
(*ii*) *if* $\Box\top \wedge \Box\bot \in$ S, S *is a sublogic of the unit logic,*
(*iii*) *if* $\neg\Box\top \wedge \Box\bot \in$ S, S *is a sublogic of the complement logic,*
(*iv*) *if* $\neg\Box\top \wedge \neg\Box\bot \in$ S, S *is a sublogic of the zero logic.*

Now, the present author at one time thought it a routine matter to eliminate the various hypothesis in parts (i)–(iv) of Thm. 2.3, and establish that any consistent congruential modal logic was a sublogic of at least one of: the identity logic, the unit logic, the complement logic, or the zero logic. This would then show, of course, that the only contra-classical congruential modal logic was the inconsistent logic. The reasoning was faulty, however, and since it is instructive to locate the fault, we reproduce it here.

LEMMA 2.4 *Every consistent modal logic has a consistent extension containing one of the four formulas mentioned in parts (i)–(iv) of Thm. 2.3.*

Proof. Note that the disjunction of the four formulas concerned is a substitution-instance of a tautology (namely $(p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$), it belongs to all modal logics, and so to an arbitrarily selected consistent such logic S. Calling the four disjuncts (i.e., $\Box\top \wedge \neg\Box\bot$ etc.) D_1, D_2, D_3, D_4, consider the smallest modal logic, S + D_i as we shall call it, extending S and containing D_i, for $i = 1, \ldots, 4$. In each case, if S + D_i is inconsistent, then $\neg D_i$ is provable in S. (This follows from the fact that D_i contains no propositional variables, so the smallest uniform-substitution-and-*Modus-Ponens*-closed superset of S containing D_i is the smallest *Modus-Ponens*-closed superset of S containing D_i. Thus if this set is inconsistent, there is derivation of $\bot$ using a finite number of applications of *Modus Ponens* from S $\cup$ $\{D_i\}$, and hence such a derivation of $\neg D_i$ from S.) If each $\neg D_i \in$ S, however, then S itself would be inconsistent, contrary to our hypothesis, which means that at least one of the extensions S + D_i is consistent. ◗

So far, so good. We continue with the tempting but erroneous line of reasoning which would purport to show that no consistent congruential modal logic is contra-classical. Take a consistent congruential modal logic S, and consider the four extensions S + D_i of Lemma 2.4, at least one of which, the Lemma tells us, is a consistent extension of S. We then appeal to Theorem 2.3 to conclude that we are dealing in each of the four cases with one of the logics in which $\Box$ can be replaced by a 1-ary operator defined in terms of our boolean primitives, to turn its theorems into tautologies. We pause for a paragraph break here so that the reader can reflect as to where the fallacy has occurred in this reasoning.

The answer is (of course) that to appeal to Theorem 2.3 we need to know that the logics S + D_i are themselves congruential, whereas our initial hypothesis was simply that S was (consistent and) congruential. Since congruentiality is not guaranteed to be preserved on passage from a logic to its extensions, we either need to supply a special justification for its being preserved in the present instance, or to think again – either attempting a different a different proof-strategy altogether, or reconsidering whether it is indeed true that no consistent congruential modal logics are contra-classical. In the following section, we shall see that just such reconsideration is indeed required. Under the rubric of 'alternative proof-strategies', we might think to secure the congruentiality of the logics $\mathsf{S} + \mathrm{D}_i$ by understanding them instead to be the least congruential modal logics $\supseteq \mathsf{S} \cup \{\mathrm{D}_i\}$; but this just moves the bump under the carpet from one place to another: for what is to guarantee that these extensions are consistent?

3. The Case of Demi-Negation.

In Humberstone [1995] the possibility is considered of a 1-ary connective two applications of which are equivalent to one application of $\neg$, the latter having its accustomed classical properties. Accordingly, the connective in question is described as one expressing *demi-negation*; there it is symbolized by '§', but here, for continuity with the discussion of the preceding section, we shall use the notation '$\Box$'. (This was avoided in the paper cited because there was occasion—in pp.20–23 thereof—to consider also another singulary connective which was represented by '$\Box$' because of the greater suggestiveness of the notation in its case. It is clear from the very broad notion of modal logic defined in the preceding section that there is no association of '$\Box$' with necessity or with any other notion one might capture semantically in terms of universal quantification over some range of accessible points.) Another change we make is to consider demi-negation in the framework FMLA rather than SET-SET. We touch on the incarnation of the logic of demi-negation in the latter framework in the following section, but for the moment we want as much continuity as possible with our recent concerns.

We shall use the logic of demi-negation as an example of a consistent congruential modal logic which is contra-classical. For this purpose, it is not necessary to provide a complete syntactic characterization of the logic of demi-negation, as long as we can verify (as in the proof of Prop. 3.1 below), that certain syntactic conditions are satisfied by the logic as characterized semantically, conditions which will suffice to establish its status as an example of the desired kind. But what semantic apparatus do we have in mind? Adapting the treatment in Humberstone [1995] to something looking like a not-too-unfamiliar possible worlds setting, let a *model* from now on be a structure $\langle w, w', V\rangle$ in which V is a function assigning subsets of $\{w,w'\}$ to the propositional variables. The elements of the latter set, w and w', which we can think of informally as possible worlds, are not required to be distinct, though if they are then of course $\langle w,w',V\rangle$ and $\langle w',w,V\rangle$ are distinct models. If $\mathrm{M} = \langle w,w',V\rangle$ is such a model, then truth of a formula A at an element $x \in \{w,w'\}$, which will be notated thus "$\mathrm{M} \vDash_x \mathrm{A}$", is defined inductively:

- If $\mathrm{A} = p_i$, then $\mathrm{M} \vDash_x \mathrm{A}$ iff $x \in V(p_i)$.
- If $\mathrm{A} = \mathrm{B} \wedge \mathrm{C}$, then $\mathrm{M} \vDash_x \mathrm{A}$ iff $\mathrm{M} \vDash_x \mathrm{B}$ and $\mathrm{M} \vDash_x \mathrm{C}$,
- If $\mathrm{A} = \neg\mathrm{B}$, then $\mathrm{M} \vDash_x \mathrm{A}$ iff not $\mathrm{M} \vDash_x \mathrm{B}$; and similarly for other boolean connectives.
- If $\mathrm{A} = \Box\mathrm{B}$, then $\mathrm{M} \vDash_x \mathrm{A}$ iff either ($x = w$ & $\mathrm{M} \vDash_{w'} \mathrm{A}$) or ($x = w'$ and not $\mathrm{M} \vDash_w \mathrm{A}$)

We need a notion of validity for our semantic characterization of demi-negation, so let us define a formula A to be *valid* just in case, for every model $\mathrm{M} = \langle w,w',V\rangle$, we have $\mathrm{M} \vDash_x \mathrm{A}$ for each $x \in \{w,w'\}$. The logic we are interested in we shall simply call *Demi*: it comprises precisely the formulas which are valid in the sense just defined.

PROPOSITION 3.1 *The logic Demi is a consistent congruential contra-classical modal logic.*

Proof. It is easy to see that *Demi* is a modal logic. Since p (alias p_1), for example, is clearly not a valid formula according to the above semantics, *Demi* is consistent. For congruentiality, suppose that A ↔ B is valid. Thus A and B are alike in truth-value at the first (w) element in every model, and also alike in truth-value at the second (w') element in each such model. Suppose that this is not so for the formulas □A and □B, because, say, $M \vDash_x$ □A while not $M \vDash_x$ □B. If x is w, (where M is $\langle w,w',V\rangle$), the latter means, by the semantical clause above governing □, that not $M \vDash_{w'}$ B, so our hypothesis of agreement in respect of A and B means that not $M \vDash_{w'}$ A, contradicting the assumption that $M \vDash_w$ □A. We leave the reader to obtain a similar contradiction for the case in which x is w'. For contra-classicality, we gain leave the reader to verify that the semantics makes $\Box\Box p \leftrightarrow \neg p$ valid – thereby justifying the name *Demi* for the present logic – and to check that none of the four singulary boolean connectives is available to translate □ and turn this formula into a tautology. ◗

The example of demi-negation has now served its main purpose, in showing the falsity of the speculation aired earlier to the effect that congruentiality ruled out contra-classicality (except in the uninteresting case of outright inconsistency). Before closing this section, however, we pause to notice another general moral to be drawn from the fact that the result speculated about does not hold, and then (Theorem 3.3) to extract from our discussion a general condition which is necessary and sufficient for a modal logic to be contra-classical. Although we shall call it a conjecture for convenience, the label is only provisionally appropriate since we shall be settling its status definitively. We identify a rule with the set of all its applications, the applications of an n-premiss rule (in the framework FMLA) being $(n+1)$-tuples of formulas $\langle A_1,\ldots,A_n, B\rangle$ where B is the conclusion of the application in question. Thus *Modus Ponens* has for its applications (in a given language) the triples ⟨A, A → B, B⟩ for all formulas A, B (of that language).[12] Such a rule is *substitution-invariant* if any substitution-instance of an application of the rule is an application of the rule. (Substitution-invariant rules are sometimes called "structural" rules—though of course in our discussion we are using the latter term in the unrelated Gentzen-derived sense of: rules not governing particular connectives.) We continue to work with the language of modal logic, and consider for simplicity only one-premiss rules in stating the

RULE CONJECTURE. *Let* ρ *be a substitution-invariant one-premiss rule and* S *be a consistent modal logic closed under* ρ. *Then the smallest modal logic extending* S ∪ {A → B| ⟨A,B⟩ ∈ ρ} *is itself consistent.*

For example, consider the case of ρ as the set of all pairs ⟨A, □A⟩ (A any formula), something usually called the rule of necessitation, under which all normal modal logics are closed (by the n =

[12] Why should *Modus Ponens* consist of the triples ⟨A, A → B, B⟩ rather than of ⟨A → B, A, B⟩? No reason, of course. The reader who prefers to think of n-premiss rules as ordered pairs whose first element is an n-membered multiset of formulas (the premisses) and whose second element is a formula is encouraged to do so. (It seems inadvisable to go further, however, and collect the premisses into a mere *set* of formulas, since it is attractive represent all applications of a two-premiss rule, for instance, as exhibiting a common pattern whether or not the two formula-occurrences functioning as premisses happen to be occurrences of the same formula.)

0 case of the condition (5) ⇒ (6) used at the start of Section 2 to define normality).[13] The Rule Conjecture would tell us that any consistent normal modal logic has a consistent extension to a modal logic in which the implicational formulas A → □A are all provable. This is clearly correct, since all such formulas are provable in the Post-complete normal modal logics (the identity logic and the unit logic) mentioned already. In fact, the smallest normal modal logic to contain all such formulas is precisely the intersection of those two logics. Similarly, it is well known that the smallest normal modal logic (usually called **K**) is closed under the rule ("denecessitation") whose applications are the pairs ⟨□A, A⟩; here the consistent extension promised by the Rule Conjecture is the logic **KT** (sometimes called **T**). Another example which also concerns a rule which is admissible though not derivable in any of the familiar axiomatizations of sublogics of **S5** is the rule considered in §4 of Porte [1981] whose applications are ⟨¬□¬A ∧ ¬□A, ⊥⟩, for which the Rule Conjecture delivers as a consistent extension the intersection of the unit and identity logics (determined by the class of one-element Kripke frames).[14] A handful of confirming instances do not do a lot for a conjecture purporting to cover all substitution-invariant rules (and all consistent modal logics S), however.

To bring our recent experience with demi-negation to bear on the Rule Conjecture, it will be helpful to have the following terminology available. We call a modal logic *extensional* if it contains every formula of the form (A ↔ B) → (□A ↔ □B). Below, we will say that an arbitrary 1-ary connective # is extensional according to a logic if all instances of a similar schema (with # occupying the position of □ here) are provable in the logic. Note that S can be extensional in this sense, without there being a boolean (i.e. □-free) formula $C(p_1)$ for which $\Box p_1 \leftrightarrow C(p_1) \in$ S: S need not, that is, be identifying □ with any particular truth-functional connective. (See the discussion of truth-functionality *vs.* pseudo-truth-functionality in the following section.[15]) Alternatively put, an extensional modal logic need not be one of: the identity logic, the complement logic, the unit logic, the zero logic.[16] Nevertheless, the smallest extensional modal logic is the intersection of these four logics, so certainly, being included in none of the four, *Demi* is not extensional and has no extensional consistent extension. This decides the Rule Conjecture in the negative:

PROPOSITION 3.2 *The Rule Conjecture is false.*

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[13] The Rule Conjecture is related to Dummett's and Pogorzelski's notions (respectively) of smoothness and structural completeness, but we shall not go into the relationships here. The interested reader is referred to Wojtylak [1991], which also provides an extensive bibliography.

[14] We write "**S5**" rather than, as with S6 etc., "S5" because we are following the Segerberg–Chellas 'anatomical' policy for labelling normal modal logics according to which $\mathbf{KX}_1\ldots\mathbf{X}_n$ is the smallest normal modal logic containing all instances of the schemata labelled $\mathbf{X}_1\ldots\mathbf{X}_n$, and the prefix "**KT**" is abbreviated to "**S**" when followed by a numerical label. Indicentally, we can also consider the deliverances of the analogue of the Rule Conjecture outside of the area of modal logic as conceived here. For instance, the purely implicational Abelian *BCI* logic described in Section 1 (including note 2) is closed under the rule ("Conversion") whose applications are all pairs ⟨A → B, B → A⟩. (In fact this logic is precisely the least extension of *BCI* logic closed under this rule.) The consistent extension we obtain by following the dictates of the Rule Conjecture then adds as axioms all instances of the schema (A → B) → (B → A); the special Abelian schema (2) from Section 1 is now redundant. This gives the equivalential fragment of classical logic in FMLA (with ↔ written as →). The point about the admissibility of Conversion is restricted to the →-fragment and does not, as Meyer and Slaney [1989] explains, apply to their full system of Abelian logic with conjunction, disjunction, etc. A purely algebraic version of the point may also be found in Daoji [1987], esp. Theorem 2.

[15] Or, for the source material, Humberstone [1986], [1997*a*].

[16] In the papers cited in the preceding note, these logics are associated with the labels **I**(dentity), **N**(egation), **V**(erum) and **F**(alsum), respectively.

Proof. Take S = *Demi* and ρ = the rule of congruentiality (with applications ⟨A ↔ B, □A ↔ □B⟩) for a counterexample. The consistent extension promised by the conjecture would have to be extensional, and as remarked, *Demi* has no extensional consistent extensions. ◗

To keep the discussion self-contained rather than expecting the reader to take on trust the observations made in the sources cited in the preceding paragraph, it may be of interest to see directly how inconsistency arises when we combine the extensionality schema with the formulas in *Demi*. We will make use of the fact that all instances of the schema (¬A ↔ ¬B) → (A ↔ B) are tautologous. This is already enough to show that *Demi* is closed under not only the congruentiality rule but also the converse rule, since if □A ↔ □B ∈ *Demi*, then by congruentiality, □□A ↔ □□B ∈ *Demi*, so by the fundamental demi-negation equivalence of a double '□' with a single '¬', we have ¬A ↔ ¬B ∈ *Demi* and hence A ↔ B ∈ *Demi*. By similar reasoning, if the extensionality schema is added to *Demi*, it yields its own converse, since extensionality gives (□A ↔ □B) → (□□A ↔ □□B), whence via the fundamental demi-negation equivalence we again get (□A ↔ □B) → (A ↔ B). Thus in *Demi* with the extensionality principle added we in fact have

(9) (A ↔ B) ↔ (□A ↔ □B)

and hence as a special case

(10) (⊤ ↔ □⊥) ↔ (□⊤ ↔ □□⊥)

The left-hand side simplifies by truth-functional reasoning to □⊥, and the right-hand side similarly, after □□⊥ is replaced by ¬⊥, to □⊤. Thus we have

(11) □⊥ ↔ □⊤

from which the converse of the congruentiality rule, already remarked on, we then obtain (12)

(12) ⊥ ↔ ⊤

revealing the inconsistency of the envisaged extension.

More generally (see Section II of Humberstone [1986]), the condition of extensionality is equivalent, as a condition on modal logics, to the requirement that the following disjunction be provable in the logic:

(13) (□*p* ↔ *p*) ∨ (□*q* ↔ ¬*q*) ∨ (□*r* ↔ ⊤) ∨ (□*s* ↔ ⊥)

Each disjunct of (13) can be thought of as equating □ with one of the four 1-ary boolean connectives, which is why we have written the last two disjuncts as they are and not, more concisely, as □*r* and ¬□*s*, respectively. (In fact, if were really being careful on this point, we would write these disjuncts not as they appear above, but as □*r* ↔ $\top_1$ *r* and □*s* ↔ $\bot_1$ *s*, where in the latter case '$\bot_1$ A' can be thought of as abbreviating '¬$\top_1$ A'. $\top_1$ is of course just the *n* = 1 case of the $\top_n$ defined immediately before Proposition 1.1.)

The disjunctive characterization (13) of extensionality is useful for giving the general upshot of our discussion in this section

THEOREM 3.3 *For any modal logic* S, S *is contra-classical if and only if Ext*(S) *is inconsistent, where Ext*(S) *is the smallest extensional modal logic extending* S.

Proof. *'If':* For any translation τ, $\Box^\tau$ is extensional according to CL, so τ cannot map all theorems of S to tautologies if *Ext*(S) is inconsistent.

'Only if': Call the disjuncts of (13) E_1, E_2, E_3, E_4. Since these disjuncts have no propositional variables in common with each other, *Ext*(S) is consistent only if at least one of S + E_i is consistent (i = 1,...,4). (See Lemmon [1966]; the '+' notation is to be understood as in the proof of Lemma 2.4.) Assume then that *Ext*(S) is consistent, and pick E_i for which S + E_i is consistent. This gives a suitable classicizing translation τ for S by telling us what $\Box^\tau$ should be: e.g., if $i = 4$, take $\Box^\tau$ as $\bot_1$. ◗

4. Other Frameworks

In this section we consider contra-classicality in SET-FMLA and SET-SET, these being the 'other frameworks' of our title.

In Proposition 1.1 we saw that for the framework FMLA, there were no consistent contra-classical logics. In SET-FMLA, it may appear that we first need to decide exactly what we mean by a logic before examining the corresponding question. Similarly for SET-SET. We know that we are to think of logics as sets of sequents of the respective frameworks, and continue to require closure under uniform substitution, but this does not settle exactly what is to be required in the way of structural rules – rules that is that are formulable without reference to any particular connectives. In the case of SET-FMLA, we have the rules ($\mathbb{R}$), ($\mathbb{M}$), and ($\mathbb{T}$) – to use labels adapted from Scott [1974], and with the usual notational liberties being taken ("$\Gamma, A \succ B$" for "$\Gamma \cup \{A\} \succ B$", etc.):

($\mathbb{R}$) $\quad A \succ A$

($\mathbb{M}$)
$$\frac{\Gamma \succ B}{\Gamma, A \succ B}$$

($\mathbb{T}$)
$$\frac{\Gamma, A \succ B \qquad \Gamma \succ A}{\Gamma \succ B}$$

These rules guarantee that when we define a relation $\vdash$ in terms of any collection of sequents (of SET-FMLA over some fixed language) proof-system closed under them by: $\Gamma \vdash A$ iff for some finite $\Gamma_0 \subseteq \Gamma$, the sequent $\Gamma_0 \succ A$ is provable, then $\vdash$ is a finitary consequence relation with the property that for finite Γ_0, we have $\Gamma_0 \vdash A$ just in case $\Gamma_0 \succ A$ is provable. All consequence relations under consideration here are finitary, and we will not explicitly include this qualification. The corresponding structural rules for SET-SET will not be given here (they can be recovered from Scott [1974]); they enjoy a similar relationship with the generalized consequence relations mentioned in Section 1. Profoundly contra-classicality for logics conceived of as sets of sequents of SET-FMLA (SET-SET) closed under uniform substitution and the three structural rules mentioned above – or equivalently, finitary substitution-invariant (generalized) consequence relations – can be understood in the same way as for FMLA in Section 1, abstracting away from the choice of logical framework. That is, a (generalized) consequence relation $\vdash$ is profoundly contra-classical just in case there is no translation τ from the language of $\vdash$ to the language of the classical logic (with, say, K_{bool} as set of connectives) such that for every sequent $\sigma \in \vdash$,[17] $\tau(\sigma) \in \vdash_{CL}$, where for $\sigma = A_1,\ldots,A_m \succ B$ (for $\sigma = A_1,\ldots,A_m \succ B_1,\ldots,B_n$), $\tau(\sigma)$ is $\tau(A_1),\ldots,\tau(A_m) \succ \tau(B)$ (respectively, $\tau(A_1),\ldots,\tau(A_m) \succ \tau(B_1),\ldots,\tau(B_n)$). Here by $\vdash_{CL}$, we mean, as appropriate, the consequence relation or generalized consequence relation associated with classical logic.[18] As in FMLA, we can relax the 'profundity' selectively by insisting that candidate translations τ are fixed on some set K of connectives of the language of $\vdash$, giving a notion of contra-classicality *modulo K*. Because of the greater structural resources of SET-FMLA and, especially, SET-SET, the need to do so is less pressing that in FMLA. For example, we can redescribe the demi-negation example of the preceding section wholly purged of the boolean connectives in SET-SET; "consistent" means that not all sequents of the framework in question are provable:

PROPOSITION 4.1 *The smallest logic in SET-SET, in a language with a single 1-ary connective* $\Box$, *containing all sequents of the forms (a) and (b), is profoundly contra-classical (though consistent)*:

(*a*) $A, \Box\Box A \succ \varnothing$ (*b*) $\varnothing \succ A, \Box\Box A$

In Humberstone [1995] the logic of demi-negation was systematized in SET-SET by taking, alongside the structural rules and also a congruentiality rule for '$\Box$' with premiss-sequents $A \succ B$ and $B \succ A$ and conclusion sequent $\Box A \succ \Box B$. We do not need to mention this for Prop. 4.1, however, since any would-be $\Box^\tau$ will have to be congruential (according to CL – to use this phrase in the same way as we used 'extensional according to' above). As for the structural rules ($\mathbb{R}$), ($\mathbb{M}$), and ($\mathbb{T}$), we shall from now on assume for the sake of the connection with consequence relations and generalized consequence relations, that any SET-FMLA or SET-SET logic must be closed under them to qualify as a logic.

It is interesting what happens to the example in SET-FMLA. In general, to obtain a SET-FMLA logic sound and complete w.r.t. the same class of valuations as a given SET-SET logic, empty right-hand sides should be replaced by a (new) schematic formula variable, and multiple right-hand sides by a rule trading them in for "common consequences", procedures which convert (*a*) and (*b*) into (*a*′) and (*b*′):

(*a*′) $A, \Box\Box A \succ B$ (*b*′)
$$\frac{\Gamma, A \succ C \qquad \Gamma, \Box\Box A \succ C}{}$$

[17] Recall from Section 1 that we regard '$\Gamma \succ A$' ('$\Gamma \succ \Delta$') as a notation for the ordered pair $\langle\Gamma, A\rangle$ (for $\langle\Gamma, \Delta\rangle$), so, since $\vdash$ is a set of such pairs, the "$\sigma \in \vdash$" formulation in the text is appropriate. (Of course one would never write (e.g.) "$\Gamma \succ A \in \vdash$", since one can say this more economically as $\Gamma \vdash A$.)

[18] By a *valuation* we mean a function from the set of formulas assigning each of them a truth-value (T or F); if v is such a function and it does not violate the conditions imposed by the usual truth-table account of the meanings of the connectives in K_{bool}, we call *v* a *boolean* valuation. (More specifically, for a given $\# \in K_{bool}$, we call *v* #-boolean if *v* abides by these conditions at least for #-compounds; thus *v* is $\to$-boolean, for example, provided that for all formulas A, B, we have $v(A \to B) = F$ iff $v(A) = T$ and $v(B) = F$.) A generalized consequence relation $\vdash$ is *determined by* a class of valuations just in case for all Γ, Δ, we have $\Gamma \vdash \Delta$ iff for no valuation in the class assigns T to every element of Γ and F to every element of Δ. The generalized consequence relation, $\vdash_{CL}$, associated with classical logic is then just the generalized consequence relation determined by the class of all boolean valuations. If $\sigma \in \vdash_{CL}$, we sometimes call σ *tautologous*; this means that σ holds on every boolean valuation, where in general $\Gamma \succ \Delta$ *holds* on *v* provided *v* does not assign T to all $A \in \Gamma$ and F to all $B \in \Delta$. A valuation *v* is *consistent with* a consequence relation $\vdash$ when every $\sigma \in \vdash$ holds on *v*. All concepts introduced in this note are to be taken to apply to consequence relations and sequents of SET-FMLA by restricting to the case in which Δ contains exactly one element.

$$\Gamma \succ C$$

To think of (a') and (b') as two parts of a condition on how '$\Box$' is to behave, since (b') is a sequent-to-sequent rule with two (sequent-)premisses, we should think of (a') as also being a sequent-to-sequent rule: a zero-premiss rule. In effect, our discussion up to this point has only considered zero-premiss rules (or individual formulas which are the 'applications' – i.e., instances – of such rules in FMLA) as making for contra-classicality and while we do not intend to depart from this, some clarificatory comments are in order. (Those who feel no need of such further clarification may omit the following paragraph – which makes use of terminology defined in note 19 – without loss of continuity.)

At first sight, one would expect to be faced with a distinction to have to make in deciding whether a collection of rules deserved to be called contra-classical. Option (1): a set of rules is contra-classical just in case there is no translation τ (to the language of classical logic) such that for every application, from, say, $\sigma_1,\ldots,\sigma_n$ to σ_{n+1} of a rule (here assumed to be an n-premiss rule) in the set, the transition from $\tau(\sigma_1),\ldots,\tau(\sigma_n)$ to $\tau(\sigma_{n+1})$ preserves the property, for an arbitrary boolean valuation, of holding on that valuation. Option (2): as with option (1) except that property whose preservation in the transition from $\tau(\sigma_1),\ldots,\tau(\sigma_n)$ to $\tau(\sigma_{n+1})$ is not that of holding on an arbitrarily selected boolean valuation but the property of holding on all of them (i.e. tautologousness). We do not, however, at least for substitution-invariant sequent-to-sequent rules, have to face a decision here, since a well known fact about such rules is that they have the local (valuation-by-valuation) preservation characteristic just in case they have they have the global preservation characteristic (preservation of tautologousness): this is a special fact about boolean valuations, and it means that Options (1) and (2) coincide for the rules whose applications are the 'translations' $\langle\tau(\sigma_1),\ldots,\tau(\sigma_n),\tau(\sigma_{n+1})\rangle$ of the applications of the rules whose contra-classicality is at issue. Notice, however, that it is not logics but collections of rules that are now being classified as contra-classical or otherwise. This would be more of a shift than we had in mind for the present discussion, however, the intention being merely to move from logics as collections of sequents of the framework FMLA (i.e., for all intents and purposes, collections of formulas) to logics as collections of sequents of SET-FMLA and SET-SET (or the associated consequence relations and generalized consequence relations). For this reason, we shall not further consider the matter of contra-classicality for collections of sequent-to-sequent rules (excepting the case of 0-premiss such rules, of course), beyond making one further remark. One might start with the collection of sequents and then consider all the rules under which this collection is closed: that would give a definite set of rules to consider without privileging any one proof-system over another (where both proof-systems, considered as sets of primitive rules, both yield precisely the same provable sequents): compare the 'general' notion of Post-completeness mentioned in note 10 (where the framework was FMLA). Consider in particular the pure negation fragment of **CL** in SET-SET. The class of all such tautologous SET-SET sequents in which no connective other than $\neg$ appears is easily seen to be closed under the rule (14), in which $\neg\Gamma = \{\neg C \mid C \in \Gamma\}$ (and similarly $\neg\Delta$):

(14) $$\frac{\Gamma \succ \Delta}{\neg\Gamma \succ \neg\Delta}$$

and this is contra-classical rule on either of the semantic characterizations given above (as Option 1 and Option 2), or in the more syntactical sense that full **CL** (as opposed to the negation fragment) is not closed under the rule. Now, in reply to the suggestion that we have here slipped back into the superficial notion of contra-classicality – overlooking the possibility of reconstruing via a suitable $\neg^{\tau}$, the occurrences of '¬', we can alter the example from (14) to the structural rule (15) under which the set of sequents in question is also closed, and there is no scope for re-interpretation:

(15) $$\frac{\Gamma \succ \Delta}{\Delta \succ \Gamma}$$

Concerning this situation we remark only that it would be intolerable to have a notion of contra-classicality according to which a proper sublogic of CL turned out to be contra-classical. Recall we are concerned with deviation by commission! (Of course this still allows room for the development of a notion of contra-classicality for proof-systems or collections of rules, and for the verdict that any such system for which (14) and (15) were not only admissible but actually derivable – primitive or derived – was indeed contra-classical. It would after all *not* be a subsystem of a proof-system for (full) CL.)

Steering clear, then, of rule-based conceptions of contra-classicality, we still have the question left open as to whether our SET-FMLA logic with (a') and (b') above is contra-classical in our sequent-based sense. An answer requires that we inspect not the rule (b') but sequents that this rule, along with (a') and the structural rules ($\mathbb{R}$), ($\mathbb{M}$), and ($\mathbb{T}$), renders provable. It is not hard to turn up examples of provable sequents which show that the logic is indeed contra-classical. For (a') and (b') are, after all, just rules which suffice, in the company of the structural rules, and when the doubled '□'s are replaced by '¬'s, for the pure negation fragment (this time in SET-FMLA rather than, as lately considered, in SET-SET) of CL. Thus using the rules, we can obtain a proof of both the sequents listed under (16):

(16) $p, \Box\Box p \succ q$ $\qquad\qquad$ $p \succ \Box\Box\Box\Box p$

The first sequent is just an instance of rule (a'), while the second is derivable by an appeal to (b') from another application of (a') – the sequent $p, \Box\Box p \succ \Box\Box\Box\Box p$ – and the {($\mathbb{R}$), ($\mathbb{M}$)}-provided sequent $p, \Box\Box\Box\Box p \succ \Box\Box\Box\Box p$. (Here the formula $\Box\Box p$ plays the role of A in our schematic formulation of the rule (b').) But it is easy to check that none of the four 1-ary definable boolean connectives can translate □ in the two sequents of (16) and yield tautologous results.

In Section 2, we noticed that any contra-classical modal logic (in the sense in which we were using the phrase there) had at least one translation-resistant provable formula bearing witness to this contra-classicality – thanks essentially to the availability of a conjunction connective (with the expected logical properties). In the case of the present SET-FMLA incarnation of *Demi*, with rules (a') and (b'), the contra-classicality cannot similarly be localized: there is no sequent σ provable in this system for which no translation τ into the language of CL can be found with τ(σ) tautologous. If □ appears on the left of the '≻' we can take $\Box^{\tau}$ as $\bot_1$ and if it appears on the right we can take $\Box^{\tau}$ as $\top_1$; if it appears on the left and also on the right we may make either of these choices. The result is that we have a τ with τ(σ) tautologous, because, there being no other connectives in the language, appearing on the right or left is a matter of □'s being the main connective of some formula on the right or left. We have not quite established our claim about unlocalizability, though. For we have not considered the possibility that there is some provable sequent σ in which □ does not appear at all. In the case of such a σ, τ(σ) = σ, so there would be trouble for our argument if a provable □-free sequent were not tautologous. Consideration of this possibility will also enable us to address the question of the availability of anything along the lines of Prop. 1.1 for the current frameworks. The matter is most conveniently discussed in terms of consequence relations (and indeed generalized consequence relations) rather than sets of provable sequents, so we need to bear in mind the transfer of terminology that is possible because of he one-to-one correspondence between a collection of SET-FMLA sequents closed under uniform substitution and the structural rules, on the one hand, and a finitary substitution-invariant consequence relation on the other. We say, making use of some definitions in note 19, that a consequence relation or generalized consequence relation $\vdash$ is *constant-valued* just in any valuation consistent with $\vdash$ is constant (i.e., is a constant function from formulas to {T,F}); there are two such valuations: that assigning T to every formula and that assigning F to every formula. As an immediate consequence of this definition, we have the following:

PROPOSITION 4.2 *The following assertions about a substitution-invariant (generalized) consequence relation* $\vdash$ *are equivalent:*

(*i*) $\vdash$ *is constant-valued.*
(*ii*) *For all formulas* B*,* C*, we have* B $\vdash$ C.
(*iii*) $p \vdash q$
(*iv*) *This is not the case: for all sets* $\Gamma \cup \{C\}$ *in whose formulas no connectives appear,* $\Gamma \vdash C$ *implies that* $\Gamma \vdash' C$ *for every consequence relation* $\vdash'$ *(equivalently: implies that* $C \in \Gamma$*).*

Note that the connective-free formulas alluded to in (iv) are simply various propositional variables; we cannot replace the reference to them by one to atomic formulas, because of nullary connectives. We give the 'negative' formulation of (iv) because if we remove the negation, we have a useful point about the *non*-constant-valued consequence relations: they are alike in their connective-free fragments. For the application to generalized consequence relations, the references 'C', '{C}' could be replaced by 'Δ' (and the '$C \in \Gamma$' in parenthetical comment under (iv) should then read '$\Gamma \cap \Delta \neq \varnothing$'). The main difference between the consequence relation (SET-FMLA) and the generalized consequence relation (SET-SET) cases is on the number of constant-valued logics. In SET-FMLA there are only, for any given language, two such logics: the inconsistent logic, containing every sequent, and the logic containing every sequent which has at least one formula on the left. In SET-SET, there are the inconsistent logic, the logic answering to the description just given (which in this framework is the smallest SET-SET logic containing $p \succ \varnothing$), the converse or dual (smallest logic containing $\varnothing \succ q$), and the intersection of these two (smallest logic containing $p \succ q$).

For the consequence relation we have been considering, that associated with the {(*a*′),(*b*′)} proof-system, it is clear from the semantics given for *Demi* in the preceding section (adapted to the present setting by deeming a sequent $\Gamma \succ A$ to be valid when for every model $M = \langle w,w',V\rangle$, with $M \models_w C$ for each $C \in \Gamma$, we have $M \models_w A$, and likewise in the case of *w*′), that $p \succ q$ is not provable (since it is not valid). So from the equivalence of (iii) and (iv) under Prop. 4.2, whenever a connective-free (in the present case □-free) sequent σ is provable, we have $\sigma \in \vdash'$ for all consequence relations and in particular for $\vdash' = \vdash_{CL}$. This fills the gap in the argument that the contra-classicality of our current consequence relation is not 'localizable', showing that in the case of □-free case σ, τ(σ), which for any translation τ is just σ itself, is tautologous.

We have not introduced the concept of constant-valuedness into the discussion just for the sake of making this observation about non-localizability, however. We need to have it available for a more general discussion of the fate of considerations such as those in the proof of Proposition 1.1 – according to which there are no consistent contra-classical logics in FMLA – in the more general settings of SET-FMLA and SET-SET, for each of which frameworks we have now seen the corresponding assertion to fail. The key idea was the '$\top_n$ trick', in which translated any n-ary connective by the $\top_n$, guaranteeing the tautologousness of any provable formula save for the case of a propositional variable, a case the assumption of consistency assumption allowed us to ignore. What goes wrong with this strategy when we apply it to the case of SET-FMLA, where there is, after all, always one formula on the right? What goes wrong is that this formula on the right can be a propositional variable without the logic being inconsistent. Suppose there is, in the language of the logic under scrutiny, a ternary connective # for which the logic proves $\sigma = \#(p,q,r) \succ q$, for example. Clearly we do not have $\tau(\sigma)$ tautologous when $\#^\tau$ is $\top_3$. For the strategy of (the proof of) Proposition 1.1 to work for all connectives, the consequence relation will have to satisfy a very strong condition:

(16) *For any set* Γ *and propositional variable* p_i: $\Gamma \vdash p_i \Rightarrow p_i \in \Gamma$.

One has to search far and wide (or deep and low) for a naturally occurring consequence relation meeting this condition; one example would be the pure negation restriction of the consequence relation associated with Johansson's *Minimalkalkül*. What of the cases in which (16) is satisfied, though? It turns out that consistency is not enough to secure that the translation which uses the $\top_n$ trick across the board (as for Prop. 1.1) embeds the logic in SET-FMLA **CL**. For the logic may already contain, e.g. the sequent $p \succ q$, without being inconsistent, and no amount of re-interpreting of connectives will render this tautologous. Non-constant-valuedness is what we need here, rather than just consistency. But we do not have to include this as a separate assumption, since if $\vdash$ satisfies (16), it cannot be constant-valued – see (iii) of Prop. 4.2.

PROPOSITION 4.3 *If a substitution-invariant consequence relation* $\vdash$ *satisfies the condition (16), then* $\vdash$ *is not contra-classical; in particular, for the translation* τ *given by:* $\#^\tau = \top_n$ *for each n-ary connective #, we have* $\tau(\sigma)$ *tautologous whenever* $\sigma \in \vdash$.

In view of the severity of condition (16), however, we may at this point one might consider the switch from the $\top_n$ trick to a '$\bot_n$ trick', where the definition of the n-ary $\bot_n$ can safely be left to the reader (who has already met the $n = 0$ and $n = 1$ cases). This strategy will run into difficulties if we have cases in which a #-formula (where we continue to assume # is a ternary connective) appears on the right, as it does if $q \vdash \#(A,B,C)$, for some formulas A, B, C, or even more dramatically if $\varnothing \vdash \#(A,B,C)$. Here a condition no less demanding than (16) would obviate the difficulties:

(17) *Whenever* $\Gamma \vdash A$ *and every formula in* Γ *is a propositional variable, we have* $A \in \Gamma$.

Again, satisfying this condition rules out, *inter alia*, being constant-valued.

PROPOSITION 4.4 *If a substitution-invariant consequence relation* $\vdash$ *satisfies the condition (17), then* $\vdash$ *is not contra-classical; in particular, for the translation* τ *given by:* $\#^\tau = \bot_n$ *for each n-ary connective #, we have* $\tau(\sigma)$ *tautologous whenever* $\sigma \in \vdash$.

Analogues of (16) and (17) which enable versions of Propositions 4.3 and 4.4 to go through for generalized consequence relations in place of consequence relations are provided by (18) and (19), respectively:

(18) *Whenever* $\Gamma \vdash \Delta$ *and each* $D \in \Delta$ *is a variable, then* $\Gamma \cap \Delta \neq \varnothing$.

(19) *Whenever* $\Gamma \vdash \Delta$ *and each* $C \in \Gamma$ *is a variable, then* $\Gamma \cap \Delta \neq \varnothing$.

Since such conditions are so stringent, one might consider the possibility of a mixed $\top_n$-and-$\bot_n$ strategy, hoping to treat the different connectives differently depending as form compounds having propositional variables as consequences or compounds which are themselves consequences of variables (to oversimplify the situation somewhat). The trouble is that the usual situation is that a single connective forms compounds of both kinds, even for logics which are easily seen not to be contra-classical, so rather than pursue this line of investigation further, we turn to adapting what we learnt about extensionality and contra-classicality in the preceding section (Theorem 3.3).

To do that, we need to give a pure definition of extensionality, purged of the involvement of the various boolean connectives and their classical behaviour that was part of our treatment of the extensionality of '$\Box$' according to a modal logic. It is best to present this idea initially for a generalized consequence relation $\vdash$. For illustrative purposes, we will suppose that we are dealing with a ternary connective # in the language $\vdash$. # is extensional according to $\vdash$ if the following 8 ($= 2^3$: for arity k there will be 2^k) conditions are satisfied, for all formulas $A_1, B_1, C_1, A_2, B_2, C_2$:.

(20) $A_1, A_2, B_1, B_2, C_1, C_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2)$

(21) $A_1, A_2, B_1, B_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), C_1, C_2$

(22) $A_1, A_2, C_1, C_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), B_1, B_2$

(23) $B_1, B_2, C_1,C_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), A_1, A_2$

(24) $A_1, A_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), B_1, B_2, C_1, C_2$

(25) $B_1, B_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), A_1, A_2, C_1, C_2$

(26) $C_1, C_2, \#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), A_1, A_2, B_1, B_2$

(27) $\#(A_1,B_1,C_1) \vdash \#(A_2,B_2,C_2), A_1, A_2, B_1, C_1, B_2, C_2$

The condition (20) imposes on a valuation consistent with $\vdash$ is that if it assigns the truth-values T, T, T to the three components A_1, B_1, C_1, of a compound $\#(A_1,B_1,C_1)$, and those same truth-values to the components A_2, B_2, C_2, of another compound. $\#(A_2,B_2,C_2)$, then if it also assigns the value T to one of these compounds it should assign T to the other. (21) demands that if a consistent valuation assigns the truth-values T, T, F, respectively, to the components A_1, B_1, C_1, and also to A_2, B_2, C_2, then it should again assign T to the #-compound in the one case if it does so in the other. And so on through the remaining conditions. Thus what they say collectively is that like cases (cases alike in respect of the values of the components) should be treated alike (in respect of the compound's value). If we were working in FMLA and also restricting our attention to valuations which are (in the sense explained in note 19) #-boolean for each $\# \in \{\wedge, \to, \leftrightarrow\}$, then we could wrap all of (21)–

(27) into a single formula schema:

(28) $((A_1 \leftrightarrow A_2) \wedge (B_1 \leftrightarrow B_2) \wedge (C_1 \leftrightarrow C_2)) \rightarrow (\#(A_1,B_1,C_1) \leftrightarrow \#(A_2,B_2,C_2))$

in the sense that in the class of valuations to which we have restricted ourselves, those consistent with a $\vdash$ satisfying (20)–(27) are precisely those verifying all instances of (28). In the definition given in Section 3 of the extensionality of $\Box$ according to a modal logic, we used a formulation of this kind (except that because $\Box$ is 1-ary, we only had what here appears as the A_i part of the story, without the B_i and C_i). We take it that this 3-ary example has made it clear in the general case what is meant by saying that a connective is extensional according to a generalized consequence relation, and from now on, '#' reverts to its use as an all-purpose schematic symbol for an arbitrary connective.

An explicit the semantic description of extensionality requires the idea of associating a truth-function (function from $\{T,F\}^n$ to $\{T,F\}$ for some n) with a connective. So we say that a valuation v *associates* the n-ary truth-function f with the n-ary connective # just in case for all formulas $C_1,\ldots,C_n$:

$$v(\#(C_1,\ldots,C_n) = f(v(C_1),\ldots,v(C_n)).$$

Then we say (as usual) that # is *truth-functional* with respect to a class of valuations **V** when (29) is satisfied and (less familiarly) that # is *pseudo-truth-functional* w.r.t. **V** when (30) is satisfied:

(29) $\exists f \forall v \in \mathbf{V}$: *v associates f with #* (30) $\forall v \in \mathbf{V}\ \exists f$: *v associates f with #*

where in both cases "$\exists f$" means: there exists an n-ary truth-function f. Note that a valuation may associate more than one truth-function with a given connective, but only if it is one of the two constant valuations. The main observation[19] to make about the semantics of extensionality is then concisely stateable once we avail ourselves of the abbreviation '$Val(\vdash)$' to denote the class of all valuations consistent with (the generalized consequence relation) $\vdash$:

PROPOSITION 4.5 *A connective # of the language of a generalized consequence relation $\vdash$ is extensional according to $\vdash$ if and only if # is pseudo-truth-functional w.r.t. $Val(\vdash)$.*

We are now in a position to adapt Theorem 3.3 to the present more general setting. For a generalized consequence relation $\vdash$, let $Ext(\vdash)$ be the least generalized consequence relation extending $\vdash$, according to which every connective in the language of $\vdash$ is extensional. Note that if $\vdash$ is substitution-invariant, then so is $Ext(\vdash)$.

THEOREM 4.6 *Let $\vdash$ be a substitution-invariant generalized consequence relation. Then $\vdash$ is contra-classical if and only if $Ext(\vdash)$ is constant-valued.*

Proof. *'If'*: Suppose $\vdash$ is not contra-classical, with a view to showing that $Ext(\vdash)$ is not constant-valued. By the supposition, we have a translation τ with $\tau(\sigma) \in \vdash_{CL}$ for every $\sigma \in \vdash$. Let $\vdash^{\tau}$ be the generalized consequence determined by the class of all valuations which are $\#^{\tau}$-boolean, with the connectives $\#^{\tau}$ treated as new primitive connectives, for each # of the language of $\vdash$; it does no harm to regard them as additional (redundant) primitives in the language of $\vdash_{CL}$, in which case we

[19] See Humberstone [1986] for additional details, though the terminology here is somewhat different (and matches that used in Humberstone [1997*b*], which also includes further remarks on related terminology in the literature).

have $\vdash^{\tau} \subseteq \vdash_{CL}$. Since every connective in the language of $\vdash_{CL}$ (enriched or otherwise) is extensional according to $\vdash_{CL}$, $\vdash_{CL}$ is a non-constant-valued extensional extension of $\vdash^{\tau}$. Finally, rewriting every $\#^{\tau}$ back as #, and calling the re-notated generalized consequence relation obtained from $\vdash^{\tau}$ by this process $\vdash^{+}$, $\vdash^{+}$ is a non-constant-valued extensional generalized consequence relation $\supseteq \vdash$, *Ext*($\vdash$), by definition the smallest extensional extension of $\vdash$, cannot be constant-valued.

'Only if': Suppose *Ext*($\vdash$) is not constant-valued, with a view to showing that $\vdash$ is not contra-classical. This supposition, together with Prop. 4.5 and the remark preceding it, allows us to select a non-constant v_0 from *Val*(*Ext*($\vdash$)), which associates with each connective # of the language of *Ext*($\vdash$) (which is also the language of $\vdash$) a unique truth-function $f_{\#}$. Let $\vdash_f$ be the (substitution-invariant) generalized consequence relation determined by the class $\mathbf{V}_f$ of all $v \in$ *Val*(*Ext*($\vdash$)) which make this same association of the various $f_{\#}$ with the various # as v_0 does, and let $\mathbf{f}_{\#}$ be the (not necessarily primitive) boolean connective which expresses $f_{\#}$ in the sense of being associated with $\mathbf{f}_{\#}$ by every boolean valuation. This determines a translation τ, defined by setting $\#^{\tau} = \mathbf{f}_{\#}$ for each # of the language of $\vdash$. We claim that τ embeds $\vdash$ in $\vdash_{CL}$, showing that $\vdash$ is not contra-classical. That is, we claim that if $\Gamma \vdash \Delta$, then $\tau(\Gamma) \vdash_{CL} \tau(\Delta)$. For suppose that $\tau(\Gamma) \nvdash_{CL} \tau(\Delta)$. Then for some boolean valuation v, $v(\tau(C)) = T$ for all $C \in \Gamma$ and $v(\tau(D)) = F$ for all $D \in \Delta$. Now consider a valuation $u \in \mathbf{V}_f$ which makes the same assignment to the propositional variables as v does. (How do we know there is such a valuation in $\mathbf{V}_f$? Because the latter comprises all those valuations in *Val*(*Ext*($\vdash$)) associating f with #, and if we had, putting

$$\Theta_1 = \langle \{p_i \mid v(p_i) = T\}, \Theta_2 = \{\{p_i \mid v(p_i) = F\}\rangle,$$

$\Theta_1 \vdash_f \Theta_2$, then we should have Θ_1 *Ext*($\vdash$) Θ_2, contradicting the supposition that *Ext*($\vdash$) is not constant-valued. See the remark following Prop. 4.2 about the SET-SET of condition (iv).) Since for each connective #, u associates the same truth-function with each connective # as v does with $\mathbf{f}_{\#}$, we must have $u(A) = v(\tau(A))$ for every formula A, and so in particular, $u(C) = T$ for all $C \in \Gamma$ and $u(D) = F$ for all $D \in \Delta$. Therefore $\Gamma \nvdash_f \Delta$, which is impossible, since $\vdash \subseteq \vdash_f$ and $\Gamma \vdash \Delta$. (Alternative formulation: since $\mathbf{V}_f \subseteq$ *Val*($\vdash$), $u \in$ *Val*($\vdash$), contradicting our assumption that $\Gamma \vdash \Delta$.)

◗

We shall not look into the question of what can be done along the lines of the above result for the case of SET-FMLA (i.e., with consequence relations replacing generalized consequence relations), because of numerous expository complications – not the least of which is that Proposition 4.5 is not correct in the case of consequence relations (understanding *Val*($\vdash$) this time as the set of valuations consistent with a consequence relation $\vdash$). What follows the "if and only if" needs in their case to be replaced by "# is pseudo-truth-functional w.r.t. some class of valuations by which $\vdash$ is determined." Of course, the notion of extensionality itself needs a suitable redefinition for this to make sense (let alone be true), and for this purpose all conditions involving multiple right-hand sides (which in the case of (20)–(27), for instance, means all except for (20)) need to be replaced by *conditional* constraints on a consequence relation in the same way—by the device of common consequences"—that (*b*) of Prop. 4.1 was replaced by its SET-SET analogue (*b′*) shortly thereafter. In this way, (21), for instance, is replaced by the condition (31) or some equivalent condition on consequence relations $\vdash$:

(31) *If* $\Gamma, \#(A_2,B_2,C_2) \vdash D$ *and* $\Gamma, C_1 \vdash D$ *and* $\Gamma, C_2 \vdash D$, *then*

$$\Gamma, A_1, A_2, B_1, B_2, \#(A_1,B_1,C_1) \vdash D.$$

However, we leave the interested reader to follow the trail for consequence relations we have followed for their generalized cousins from that point, calling our own enquiries to a halt here.

5. Appendix: Contra-Intuitionistic Logics

We can investigate the existence of "contra-X" logics for choices of X other than as classical logic, by following the same definition except for replacing classical logic by another choice of X. In other words, whereas our discussion of contra-classicality has taken CL as a fixed target for translations and asked about the status of various candidate sources, we can also consider varying the target logic. By way of illustration, we devote this Appendix to looking into what happens when we choose X as intuitionistic (propositional) logic, henceforth IL. We also need to choose a framework, and SET-FMLA would appear to be a natural candidate. (FMLA is not promising, since Proposition 1.1 can be duplicated for IL. And, by contrast with CL, there is no single clear candidate to be the SET-SET incarnation of IL.[20] So we are considering a contra-intuitionistic logic to be a substitution-invariant (and finitary) consequence relation which cannot be embedded by any translation into the intuitionistic consequence relation $\vdash_{IL}$. A translation τ, exactly as before, is uniquely determined by its treatment, as $\#^{\tau}$, of the connectives # of the logic under scrutiny. So $\vdash$ is *contra-intuitionistic* just in case there is no such τ with $\{\tau(\sigma) \mid \sigma \in \vdash\} \subseteq \vdash_{IL}$. We do not have a general result to present here along the lines of Theorem 4.6, characterizing the full range of contra-intuitionistic logics. We confine ourselves to giving a particular example of a contra-intuitionistic logic which happens to raise some interesting issues about intuitionistic logic itself. (See for instance Proposition 5.7 below.) As for the question of an analogue to Theorem. 4.6, we note here only that $\vdash_{IL}$ is not an extensional consequence relation. For example, $\neg$ is not extensional according to $\vdash_{IL}$. The latter would require two conditions to be satisfied, (32) and (33), both understood as holding for all formulas A, B, C:

(32) $A, B, \neg A \vdash_{IL} \neg B$ (33) $[\Gamma, A \vdash_{IL} C$ *and* $\Gamma, B \vdash_{IL} C$ *and* $\Gamma, \neg B \vdash_{IL} C] \Rightarrow \Gamma, \neg A \vdash_{IL} C$

and while (32) is satisfied, (33) is not. (Take $\Gamma = \varnothing$, $A = \neg p$, $B = p$, $C = p \vee \neg p$.)

The consequence relation we shall show to be contra-intuitionistic in the sense introduced above, is none other than the relation of tautological consequence, $\vdash_{CL}$. Since what is to be established is close to various other results, we begin by emphasizing a distinction already drawn in Section 1 as the contrast between (5) and (6), which we shall write here with IL replacing CL, and with the more general understanding that 'S' stands for the logic in question in an unspecified logic logical framework: as a set of sequents, that is (identifying, for FMLA, a sequent with the sole formula appearing therein). The distinction then becomes that between (34) and (35)

(34) *For every sequent* σ: $\sigma \in S \Rightarrow \tau(\sigma) \in IL$

(35) *For every sequent* σ: $\sigma \in S \Leftrightarrow \tau(\sigma) \in IL$

To show that a logic S is contra-intuitionistic, we have to show the nonexistence of a translation τ satisfying (34), not just the non-existence of a τ satisfying the stronger condition (35). As far as (35) is concerned with classical logic as S, we summarize the (well known) results for the two frameworks w.r.t. which the question has been considered.

[20] For further details, see Gabbay [1981], esp. Chapter 3, comparing Theorem 6 of §1 and Theorem 5 of §2.

PROPOSITION 5.1 *Taking* S *as* CL*, then* (*i*) *for the framework FMLA, there exists a translation* τ *satisfying* (35)*; but* (*ii*) *for the framework SET-FMLA, there is no translation* τ *satisfying* (35).

Parts (i) and (ii) of Proposition 5.1 appear as Theorems 2.6.8 and 2.6.9 of Wójcicki [1988]; for (i) we use a τ most simply described by for each $\# \in K_{bool}$, $\#^{\tau}$ is the defined connective constructed from $\wedge$ and $\neg$ which is classically (though not in general intuitionistically) equivalent to #. The result then follows from Gödel's 1933 observation that in FMLA the conjunction-negation fragments of CL and IL coincide. For (ii) we sketch Wójcicki's proof, since we shall be adapting it.[21] A central role in the argument is played by the consequence relation (on the language with K_{bool} for its set of connectives) determined by Gödel's three-element matrix, which we can alternatively think of as the consequence relation determined by the two-element Kripke frame for IL (in which one of the elements is related by the partial ordering to the other). The FMLA version of this logic is famous as the strongest properly intermediate logic (i.e., logic strictly between IL and CL), and whereas in IL there are countably many pairwise non-equivalent formulas in one variable (and hence that many definable 1-ary connectives) for this intermediate logic—G_3 let's call it—the number comes down to a very manageable 6.[22] (When 'G_3' appears subscripted to a turnstile, we write it as 'G3' to avoid a proliferation of levels.) As Wójcicki points out, every such formula (where we take the sole variable occurring to be *p*) is equivalent, as one sees by induction on the complexity of formulas, to one on the following list:

(36) $p, \neg p, \neg\neg p, p \to p, \neg\neg p \to p, \neg(p \to p)$

The six representatives in (36) may seem familiar. They are representatives of the six distinct equivalence classes of the implication-negation fragment of IL itself. But in the case of IL, the connectives outside this fragment, in particular disjunction, yield new non-equivalent formulas – as they must, of course, if the total number of such formulas is to be infinite – here in G_3, the claim that every formula in just the variable *p* is represented on the list (36) means *every* formula, not just every implication-negation formula. (The conjunction of any two formulas in (36) is IL-equivalent to one of the formulas on the list. So we can also describe (36) as containing a representative of every formula in the implication-negation-conjunction fragment of IL in which *p* is the only variable to appear.) This makes G_3 an ideal environment in which to show things about all such one-variable formulas, since we have such a manageably small number of cases to consider. Wójcicki's strategy is to prove that in SET-FMLA, CL (*alias* $\vdash_{CL}$) is not, as he puts it[23] "definable in" G_3, which is to say, in our terminology and notation, that there is no translation τ with (in the style of (35) rather than (34)):

(37) *For every SET-FMLA sequent* σ: $\sigma \in \mathsf{CL} \Leftrightarrow \tau(\sigma) \in G_3$.

[21] The proof originally appeared in Tokarz and Wójcicki [1971], and (in outline) in Wójcicki [1970].

[22] This three-valued logic has been associated with, and occasionally named after, various people other than Gödel, as well as having gone under various other names. Historical details may be found in note 2 of Wroński [1971], though to these we should add the references Jankov [1963], [1968], where the characterization of G_3 as the strongest subclassical (not just: a maximal) intermediate logic may be found. (This is also reported, apparently independently, in Hanazawa [1966]. In both cases the point is expressed somewhat differently, namely that a necessary and sufficient condition for a tautologous formula to yield CL in FMLA when added as a new axiom to IL, is that the formula be invalid in the Gödel three-element matrix).

[23] In Tokarz and Wójcicki [1971], this is what put in terms of the phrase "reconstructable in", except that the authors could not make up their minds whether they preferred that spelling or the spelling "reconstructible in", so they oscillate freely between the two. At least the change of terminology to "definable in" fixes that problem.

This result (Theorem 1.8.9 of Wójcicki [1988]) will then be used to go on to show that the same negative conclusion can be drawn with IL in place of G_3 (Wójcicki's Theorem. 2.6.9)[24], by appeal to certain considerations about the Post-completeness (or "maximality" in Wójcicki's terms) of CL with which we need not concern ourselves. We too will be able to use a result like this about G_3 as a step on the way to obtaining the result we want about IL, except that for our interim result, we want to show that there is no τ satisfying even the weakening of condition (37) we get by replacing "$\Leftrightarrow$" with "$\Rightarrow$". Note that we cannot infer from the non-existence of a translation with the stronger "$\Leftrightarrow$"-property to the non-existence of a translation with the weaker "$\Rightarrow$"-property. But if we alter Wójcicki's proof slightly, rather than trying to reason from the result proved, we can get what we are after.

Wójcicki's argument concerns the candidate to be $\neg^\tau$ for a τ supposedly satisfying (37). Since we have $p, \neg p \vdash_{CL} q$, we must have, by the $\Rightarrow$ direction of (37), $p, \neg^\tau p \vdash_{G3} q$. But which of the one-variable formulas on our exhaustive list satisfy this demand on $\neg^\tau p$? Only $\neg p$ and $\neg(p \to p)$ do. Wójcicki continues ([1988], p.75) by first ruling out $\neg p$ as a candidate for $\neg^\tau p$, on the grounds that, again appealing to the $\Rightarrow$ direction of (37), since $\neg\neg p \vdash_{CL} p$, we must have $\neg^\tau\neg^\tau p \vdash_{G3} p$, which we do not have when $\neg^\tau = \neg$. Finally, with only $\neg(p \to p)$ left standing, Wójcicki argues—here appealing for the first time to the $\Leftarrow$ direction of (37)—that since $\neg p \nvdash_{CL} q$, we must have $\neg^\tau p \nvdash_{G3} q$, which rules out taking $\neg^\tau p$ as $\neg(p \to p)$.[25] But there is no need for this appeal: we can work this case 'forwards' too, e.g., by noting that since $p \vdash_{CL} \neg\neg p$, we must have $p \vdash_{G3} \neg^\tau\neg^\tau p$, which we cannot have for this choice of $\neg^\tau p$ as $\neg(p \to p)$. With this minor adjustment to Wójcicki's argument, then, we obtain the stronger result:

PROPOSITION 5.2 *There is no translation* τ *with the property that for every SET-FMLA sequent* $\sigma \in$ CL, *we have* $\tau(\sigma) \in G_3$.

In other words, classical logic stands in the "*contra* relation" to the three-valued Gödel intermediate logic (in SET-FMLA). Since IL $\subseteq G_3$, a translation τ mapping every sequent σ of classical logic to an intuitionistically provable sequent, would put all these $\tau(\sigma)$ into G_3, contradicting Prop. 5.2, which therefore gives us as a corollary

PROPOSITION 5.3 *Classical logic in SET-FMLA is contra-intuitionistic.*

We can cast the reasoning for this conclusion in a general light by saying that according to a consequence relation $\vdash$ for which we have (38) for (not necessarily primitive) 1-ary connectives O_1, O_2 in the language $\vdash$, that O_1 is a *left inverse* of O_2:

(38) *For all formulas* A: $O_1O_2A \dashv\vdash A$

[24] It is essential for this result that we are considering only 'definitional' translations, as Wójcicki calls them, namely schematic translations which translate the propositional variables by themselves. (In other words, translations in the sense of the whole of our discussion.) If we drop the condition on the propositional variables, we can have a faithful embedding of intuitionistic into classical logic (in SET-FMLA): see Prawitz and Malmnäs [1968], Theorem A and Corollary 1, p.218.

[25] Section 3 of Epstein [1990], Chapter 10, gives several further arguments using this same strategy.

Under the same conditions, we call O_2 a *right inverse* of O_1A (according to $\vdash$).[26] Using the fact that a formula is not in general intuitionistically equivalent to its double negation but that a negated formula is always intuitionistically equivalent to *its* double negation ("the Law of Triple Negation"), it is easy to see that $\neg$ has neither a left inverse nor a right inverse according to $\vdash_{IL}$, or indeed according to any congruential conservative extension of $\vdash_{IL}$, in which such putative inverses are added as new primitive connectives and required to satisfy the condition that $O_1\neg p \dashv\vdash p$ or that $\neg O_2 p \dashv\vdash p$ as the case may be.[27] Thus if we were working with the notion of a contra-intuitionistic logic *modulo* this or that set of connectives – meaning, as in the contra-classical case considered in Section 1, that we restrict attention to translations τ for which $\#^\tau = \#$ when # is in the set in question – then any logic containing, for one of its (not necessarily primitive) connectives #, both $\#\neg p \succ p$ and the converse sequent, or both $\neg\# p \succ p$ and its converse, would count as contra-intuitionistic *modulo* $\{\neg\}$. With a little extra work, however, we can avoid the need to exempt $\neg$ from the process of re-interpretation in this way. For we can show that where $\vdash\ = \ \vdash_{IL}$, (38) can only hold when each of O_1A, O_2A, is equivalent to A itself (for all A). Let us see how G_3 can be pressed into service to that end.

Let F be the set 1-ary matrix functions (in the three-element Gödel matrix) corresponding to the six formulas listed at (36), and of these, we denote that corresponding to p, the identity function, by e. We denote function composition by concatenation. Thus fg ($= f \circ g$ in the notation of note 12) is that function taking a matrix element x to the value $f(g(x))$.

LEMMA 5.4 *For all* $f, g \in F$: $fg = e \Rightarrow f = g = e$.

Proof. Suppose $fg = e$, with $f, g \in F$. Then g is injective, in the sense that we always have $g(x) = g(y) \Rightarrow x = y$ (by applying f to both sides in the antecedent). A quick check of the functions in F reveals that the only one of them that is injective is e itself; thus $g = e$. So since $fg = e$, we have $fe = e$, which implies that $f = e$ also. ◗

PROPOSITION 5.5. *For any 1-ary connectives* O_1, O_2, *definable in the language of* G_3, *if* $O_1O_2p \dashv\vdash_{G3} p$, *then* $O_2p \dashv\vdash_{G3} p$ *and* $O_1p \dashv\vdash_{G3} p$.

Proof. Immediate from Lemma 5.4. ◗

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[26] See Humberstone and Williamson [1997] for further details and background (though in FMLA), and Williamson [1998*b*] for subsequent developments. The notation "B $\dashv\vdash$ C" in (38) means: B $\vdash$ C and C $\vdash$ B. If a specific consequence relation is indicated by a subscript S, then in what follows, we append this subscript only to the forward turnstile, writing "B $\dashv\vdash_S$ C" to mean that B $\vdash_S$ C and C $\vdash_S$ B.

[27] The word "congruential" is important here. The case of so-called strong negation (*alias* constructible falsity) provides us with a conservative extension in which intuitionistic negation has a left inverse: but strong negation is not congruential according to this extension, in particular creating non-equivalent formulas out of the IL-equivalent $\neg p$ and $\neg\neg\neg p$. (See for example Gurevich [1977] for details.) In the author's opinion, the conservativeness – in the usual sense – of the extension should not endear it to any adherent of IL precisely because it does not in this way "conserve synonymy". (On this usage of Smiley's, formulas are *synonymous* according to a logic when interchangeable *salva provabilitate* in all sequents. Thus congruential logics are those in which provable equivalence secures synonymy. By conservation of synonymy we mean that formulas synonymous in the original logic remain synonymous in the extended logic; the reason this is a desideratum from the point of view of adherent of the original logic is that such formulas represent different sentences which express the same proposition – not something that should be disrupted by the introduction of additional vocabulary not figuring in the sentences concerned..)

Thus to show that a logic $\vdash$ "stands in the *contra* relation" to SET-FMLA $\mathsf{G3}$ ($\vdash_{G3}$) it suffices to find singulary connectives $\#_1$, $\#_2$, with the former a left inverse of the latter (or, we could equally well say, the later a right inverse of the former), with either one of them behaving according to $\vdash$ in some way the identity connective does not behave according to $\vdash_{G3}$. For then any translation τ will give us, in $(\#_1)^\tau$ and $(\#_2)^\tau$ connectives O_1 and O_2 of the language of $\vdash_{G3}$, to which Proposition 5.5 will apply. This gives the "general background" to Proposition 5.2 above. The adaptation of Wójcicki's argument presented the special case in which O_1 and O_2 coincided as ¬, with $\vdash$ being $\vdash_{CL}$, and the behaviout of ¬ which was not shared by the identity connective (according to $\vdash_{CL}$ or) $\vdash_{G3}$ was given by: $p, \neg p \vdash_{CL} q$. As with the derivation of Prop. 5.3 from Prop. 5.2, this will mean that as soon as we have $\#_1$, $\#_2$, mutually inverse according to $\vdash$, yet not "behaving like" (according to $\vdash$) the identity connective according to $\vdash_{G3}$, we can conclude that $\vdash$ stands in the *contra* relation to $\vdash_{G3}$) and hence, *a fortiori*, that $\vdash$is contra-intuitionistic.

In the originally planed version of this material, before it became apparent that Wójcicki's proof, using $\mathsf{G3}$, could be adapted, as above, to show classical logic in SET-FMLA to be contra-intuitionistic the idea was to show this by first showing Proposition 5.5 directly for IL. Since there are infinitely many (rather than just six) 1-place connectives definable in IL, this is a more demanding task, but, in response to a request to settle a conjecture of the author's on related matter, C. J. Ash produced (in 1994) an inductive proof, based on an analysis of the Rieger-Nishimura lattice, that no 1-ary Heyting algebra term function ("monomial") is injective with the exception of the identity function.[28] This would be the analogue of Lemma 5.4, from which we get Proposition 5.5 for $\vdash_{IL}$, rather than $\vdash_{G3}$. Interestingly, however, it turns out that we can get this result for $\vdash_{IL}$ itself *via* Prop. 5.5, without the need for any hard work. Instead of Ash's inductive argument about the Rieger-Nishimura lattice, we need only give it a glance to become convinced of what we need to make this transfer. We move out of SET-FMLA and into FMLA for this Lemma, to avoid entanglement in the question of how the notion of an intermediate consequence relation should be defined, a matter on which there is disagreement in the literature. (On a generous interpretation, any – perhaps one might add 'finitary' – substitution-invariant consequence relation $\vdash$ with $\vdash_{IL} \subseteq \vdash \subseteq \vdash_{CL}$ counts as an intermediate consequence relation, while for a narrower interpretation, we would add the further condition that $\vdash$ should satisfy the condition that $\Gamma \vdash A \to B$ whenever $\Gamma, A \vdash B$.

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[28] The conjecture was that for no *n*-ary H(eyting) A(lgebra) polynomial (term function) *f* is there an HA-polynomial *g* with (the "recovery equation") $a_1 = g(f(a_1,\ldots,a_n),a_2,\ldots,a_n)$ holding for all elements $a_1,\ldots,a_n$ in all HAs, except in the special case that *f* and *g* represent projection functions. (Specifically, projections to the first coordinate in this formulation – which is, we stress, no less general than one in which the recovery equation takes the form $a_i = g(a_1,\ldots,a_{i-1},f(a_1,\ldots,a_n),a_{i+1},\ldots,a_n)$. Thus Ash established this for the case of $n = 1$ but was reluctant, in the absence of detailed information about the structure of the free HA on two or more generators, to pursue the matter further. While complementation shows the falsity of the corresponding $n = 1$ version of the conjecture for Boolean Algebras, the $n = 2$ case is there refuted by, for example, the function we may for convenience denote by ↔ which we may take as both *f* and *g* in the recovery equation above. On the other hand, for the corresponding HA-polynomial, there is no way (in general) of recovering *a* from $a \leftrightarrow b$ together with *b*. To see this, consider the case in which $b = 0$, which makes *f* represent the (uninvertible) operation of pseudocomplementation. (Logical version of this point: there is no definable binary connective # such that for all A, B, we have $(A \leftrightarrow B)\ \#\ B \dashv\vdash_{IL} A$, while by contrast for $\vdash_{CL}$ we may take # as ↔. The latter point underlies various "unwanted language-dependence" objections in the philosophical literature, of which the most famous is perhaps David Miller's Minnesotan/Arizonan argument in Miller [1976].) The $n = 2$ version of the conjecture says that it is not just ↔ for which there is no *g* with $g((a \leftrightarrow b), b) = a$ (all *a*, *b*), but there are no *f* and *g* with $g(f(a,b),b) = a$, except in the trivial case in which $f(a,b) = g(a,b) = a$ (all *a*,*b*). Amongst other things, this makes for a striking difference between the free HA and the free BA on a given set of generators: in the latter case, we can get the same algebra by starting with a different set of generators (e.g., trading in one of the generators for its complement, or *a* and *b* for *a* and $a \leftrightarrow b$, and so on) but in the former case we would be unable to recover the original generators after any such replacement. (Thus Miller-style arguments would not be available if the logic in use were IL rather than CL.)

For the consequence relations we are particularly interested in, $\vdash_{\mathsf{IL}}$ and $\vdash_{\mathsf{G3}}$, this further condition is satisfied, but Rautenberg [1986] points out there is no guarantee of this in general.)

LEMMA 5.6 *Let* S_1 *and* S_2 *be intermediate logics in FMLA strictly weaker than* CL*, and* A(*p*) *be any formula in which the only variable to appear is p. Then* $A(p) \leftrightarrow p \in \mathsf{S}_1$ *if and only if* $A(p) \leftrightarrow p \in \mathsf{S}_2$.

Proof. By inspection of the Rieger-Nishimura lattice (e.g., as depicted on p.49 of Troelstra and van Dalen [1988][29]), one sees that identifying the equivalence class of *p*, [*p*] we may call it. with either of two elements which it is not (lattice-theoretically) less than, namely [$\bot$], [$\neg p$], results in inconsistency (identifying all the elements with each other). All the remaining elements, which (apart from [*p*] itself) are greater than [*p*], are greater than or equal to either [$p \vee \neg p$] or [$\neg\neg p$] (= [$\neg p \to p$]), and the identification of [*p*] with the first of these results again in inconsistency, while identifying it with the latter gives CL. ◗

Another way of putting Lemma 5.6 would be to say that for the formulas $A(p) \leftrightarrow p$ concerned, IL and G_3 agree, since any other subclassical intermediate logic lies between these two. Note that we do not say that IL and G_3 agree on all formulas constructed out of the single variable *p* (which they do not: e.g., $\neg p \vee \neg\neg p \in \mathsf{G}_3 \setminus \mathsf{IL}$, though there is, as we have already remarked, agreement on the $\vee$-free formulas): it is rather that they agree as to which such formulas are equivalent to *p*, summed up in the agreement in respect of the formulas $A(p) \leftrightarrow p$. As well as the restriction to this equivalential form, the restriction to formulas in one variable is also important. If we consider formulas of the form $A \leftrightarrow p$ with no restriction on A, then it is not hard to see that no two distinct intermediate logics agree on all such formulas. Restrictive though it is, Lemma 5.6 is all we need; recall (from note 5) that the singulary definable connectives spoken of in the following result just are the one variable formulas A(*p*) addressed in that lemma.

PROPOSITION 5.7 *For any 1-ary connectives* O_1, O_2, *definable in the language of* IL*, if* $O_1O_2p \dashv\vdash_{\mathsf{IL}} p$*, then* $O_2p \dashv\vdash_{\mathsf{IL}} p$ *and* $O_1p \dashv\vdash_{\mathsf{IL}} p$.

Proof. We argue this *via* IL and G_3 as logics in FMLA to which $A \leftrightarrow B$ belongs just in case $A \dashv\vdash_{\mathsf{IL}} B$ and $A \dashv\vdash_{\mathsf{G3}} B$ respectively. Thus the hypothesis we start with is that $O_1O_2p \leftrightarrow p \in \mathsf{IL}$; since $\mathsf{IL} \subseteq \mathsf{G}_3$, $O_1O_2p \leftrightarrow p \in \mathsf{G}_3$, so by Prop. 5.5, $O_1p \leftrightarrow p \in \mathsf{G}_3$ and $O_2p \leftrightarrow p \in \mathsf{G}_3$, from which by Lemma 5.6 we get the conclusion that these two formulas belong to IL, completing the proof. ◗

The logical analogue of 'Ash's Lemma' (no nontrivial Heyting algebra injective monomials) would be what is called the ("Outer") Cancellation Condition in Humberstone and Williamson [1997], satisfied by 1-ary # according to $\vdash_{\mathsf{IL}}$ just in case for any A, B, for which we have $\#A \dashv\vdash_{\mathsf{IL}} \#B$, we also have $A \dashv\vdash_{\mathsf{IL}} B$. Whether this too can be shown to hold only for the case in which $\#p \dashv\vdash_{\mathsf{IL}} p$ on the basis of elementary reasoning about those properties of $\vdash_{\mathsf{G3}}$ exposed above

[29] Or Figure 1 on p.329 of Nishimura [1960], though this will have to be turned upside down for a conventional depiction of the partial order involved.

is an interesting question, but one perhaps best deferred to an occasion on which the main topic under discussion is not contra-intuitionistic logics. We close by raising a question squarely within that area of concern. We have seen, working within SET-FMLA at least, that while intuitionistic logic is obviously not contra-intuitionistic (take the identity map as τ) classical logic is. That covers the extreme ends of the spectrum of intermediate logics, but what about what lies between? Precisely which intermediate logics are contra-intuitionistic?[30]

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