

The Consequence Relation of Tautological Entailment is Maximally Relevant: Answering a Question of Graham Priest

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Graham Priest has asked whether the consequence relation associated with the Anderson–Belnap system of Tautological Entailment,¹ in the language with connectives \neg , \wedge , \vee , and countably many propositional variables as atomic formulas, maximal amongst the substitution-invariant relevant consequence relations on this language. Here a consequence relation \vdash is said to be *relevant* just in case whenever for a set of formulas Γ and formula B , we have $\Gamma \vdash B$ only if some propositional variable occurring in B occurs in at least one formula in Γ . (It follows that relevant consequence relations are atheorematic in the sense that whenever $\Gamma \vdash B$ for some such consequence relation \vdash , $\Gamma \neq \emptyset$.) Here I write up in more detail the upshot of the conversation – returning an affirmative answer to Priest’s question – about this in the common room that Greg Restall and I were participating in last Friday [= October 6, 2006], dotting some “i”s and crossing some “t”s (and adding the odd further reflection).

We are concerned with the consequence relation \vdash_{TE} defined on the language described above by: $\Gamma \vdash_{\text{TE}} B$ iff for some $C_1, \dots, C_k \in \Gamma$ ($k \geq 1$), the conjunction of the C_i tautologically entails B in the sense of Anderson and Belnap. For the consequence relations extending \vdash_{TE} such commas on the left are always equivalent to conjunctions, so we can consider only the case in which there is a single formula A on the left. We recall that $A \vdash_{\text{TE}} B$ just in case, where $A_1 \vee \dots \vee A_m$ and $B_1 \wedge \dots \wedge B_n$ are conjunctive and disjunctive normal forms equivalent (according to \vdash_{TE}) to A and B respectively, we have $A_i \vdash_{\text{TE}} B_j$ for each i, j ($1 \leq i \leq m$, $1 \leq j \leq n$), and, further, that this is so for a given pair A_i, B_j just in case some literal (variable or negated variable) occurs as a conjunct of A_i and also as a disjunct of B_j .

Now let \vdash be some substitution-invariant consequence relation on the language of \vdash_{TE} such that $\vdash \subsetneq \vdash_{\text{TE}}$, with a view to showing that \vdash is not relevant. Since $\vdash \subsetneq \vdash_{\text{TE}}$, we have some A, B for which

¹Familiarity is assumed with this material, which can, however, conveniently be found in Chapter III of Anderson and Belnap [1].

$$(1) \quad A \vdash B, \quad \text{while} \quad (2) \quad A \not\vdash_{\text{TE}} B.$$

In view of (2) we have $A_i \not\vdash_{\text{TE}} B_j$, for some disjunct A_i in a dnf of A and conjunct B_j in a cnf of B . Thus A_i and B_j do not share a literal. Suppose (for a contradiction) that \vdash is relevant. Then, in view of (1), $A_i \vdash B_j$, and some variable occurs negated as a conjunct of A_i and unnegated as a disjunct of B_j or else unnegated as a conjunct of A_i and negated as a disjunct of B_j . We represent all variables of the former type by q_1, \dots, q_ℓ , of the latter type by $r_1 \dots r_m$, with $\lambda_1, \dots, \lambda_k$ and μ_1, \dots, μ_n for the remaining literals occurring as conjuncts of A_i and disjuncts of B_j respectively. (In other words, no variables occurring negated or unnegated amongst $\lambda_1, \dots, \lambda_k$ occur negated or unnegated amongst μ_1, \dots, μ_n , and none of the variables occurring in either of these lists are amongst $q_1, \dots, q_\ell, r_1 \dots r_m$.) Thus we have, writing λ for the conjunction of $\lambda_1, \dots, \lambda_k$ and μ for the disjunction of μ_1, \dots, μ_n :

$$\lambda \wedge q_1, \dots, q_\ell \wedge \neg r_1 \wedge \dots \wedge \neg r_m \vdash \mu \vee \neg q_1 \vee \dots \vee \neg q_\ell \vee r_1 \vee \dots \vee r_m.$$

By the substitution-invariance of \vdash , we may identify the variables r_i (we write just ‘ r ’) and similarly in the case of the q_i (writing ‘ q ’):

$$\lambda \wedge q \wedge \neg r \vdash \mu \vee \neg q \vee r.$$

(If $\ell = 0$ or $m = 0$, then q in both its occurrences, or r in both its occurrences in the above \vdash -statement, will be absent and the procedure which follows can be simplified further in an obvious way.)

Now select a propositional variable not figuring amongst the formulas represented above, s , say, and substitute $q \vee s$ for q , giving:

$$\lambda \wedge (q \vee s) \wedge \neg r \vdash \mu \vee \neg(q \vee s) \vee r.$$

Since $\vdash \supseteq \vdash_{\text{TE}}$ and

$\lambda \wedge q \wedge \neg r \vdash_{\text{TE}} \lambda \wedge (q \vee s) \wedge \neg r$, and also $\mu \vee \neg(q \vee s) \vee r \vdash_{\text{TE}} \mu \vee \neg s \vee r$, we have:

$$\lambda \wedge q \wedge \neg r \vdash \mu \vee \neg s \vee r.$$

Having got rid of the double occurrence of q , we proceed to do the same for r , this time substituting $r \wedge t$ for r (t another new variable):

$$\lambda \wedge q \wedge \neg(r \wedge t) \vdash \mu \vee \neg s \vee (r \wedge t).$$

As before (using the fact that $\vdash \supseteq \vdash_{\text{TE}}$), we conclude that

$$\lambda \wedge q \wedge \neg r \vdash \mu \vee \neg s \vee t,$$

which, finally, contradicts the assumption that \vdash was a relevant consequence relation since no variable occurs on both the right and the left of the \vdash here.

Conclusion: \vdash has no proper extensions in the same language which are both substitution-invariant and relevant.

A somewhat subtle point: Although we have used the Anderson–Belnap normal forms for \vdash -statements, we have not claimed that for an arbitrary extension \vdash^+ of \vdash_{TE} (an in particular for the \vdash we have been reasoning about) that we have $A_1 \vee \dots \vee A_m \vdash^+ B_1 \wedge \dots \wedge B_n$ if and only if for each i, j concerned, $A_i \vdash^+ B_j$ – and such a claim would not be correct. In fact, we do have the “only if” half of this claim and inspection of the above argument (in particular where, after presenting (1) and (2), it was said that “in view of (1), $A_i \vdash B_j$ ”) this half was all that was used. This is because a property of \vdash_{TE} (and many other familiar consequence relations), namely that $C \vdash_{\text{TE}} E$ and $D \vdash_{\text{TE}} E$ imply $C \vee D \vdash_{\text{TE}} E$, is not guaranteed to be preserved in passing from \vdash_0 to $\vdash_1 \supseteq \vdash_0$, even for \vdash_0, \vdash_1 on the same language (i.e., no new logical vocabulary in the language of \vdash_1).² A simple example of this is provided by the consequence relation determined by the matrix whose algebra is (that of) BN4, but with the top element as the sole designated element. For this consequence relation, which without risk of confusion with the above bearer of this name, we can denote by \vdash , we have:

$$p \wedge \neg p \vdash r \quad \text{and also} \quad q \wedge \neg q \vdash r,$$

while at the same time

$$(p \wedge \neg p) \vee (q \wedge \neg q) \not\vdash r.$$

This shows that there are more (substitution-invariant) consequence relations extending \vdash_{TE} (on the same language) than one might have initially thought. The least such consequence relation \vdash satisfying (for all formulas A, C) $A \wedge \neg A \vdash C$, as the above example shows, is strictly included in the least such consequence relation satisfying (for all A, B, C) $(A \wedge \neg A) \vee (B \wedge \neg B) \vdash C$. Of course, both of these consequence relations (extending \vdash_{TE}) fail to be relevant, as the above argument shows must be the case, confirming Priest’s conjecture, for all proper extensions of \vdash_{TE} (with the same language).

References

- [1] Anderson, A. R., and N. D. Belnap [1975] *Entailment: the Logic of Relevance and Necessity, Vol. I*, Princeton University Press, Princeton, NJ.

²A similar disjunction property not inherited by extensions is discussed in Rautenberg [2]. Unlike disjunction, conjunction displays no such anomalous behaviour amongst the extensions of a consequence relation according to which relation itself, it enjoys the accustomed properties. Such differences between \wedge and \vee are the result of the asymmetrical treatment provided by consequence relations and disappear if instead one considers *generalized* consequence relations, allowing empty or multiple right-hand sides (‘succedents’).

- [2] Rautenberg, W. [1981] '2-Element Matrices', *Studia Logica* **40**, 315–353.